Solution to Problem 33

Problem. Use the variable-elimination algorithm for checking satisfiability of 2-SAT formulas that we had in class to find the values that satisfy the following formula:

$$(a \vee b) \,\&\, (\neg a \vee b) \,\&\, (\neg a \vee \neg b) \,\&\, (a \vee \neg c) \,\&\, (\neg a \vee \neg c) \,\&\, (b \vee \neg c).$$

Solution. In this formula, we have three Boolean variables: a, b, and c. According to the algorithm, we need to eliminate them one by one.

Let us first eliminate the variable a. According to the general algorithm:

- each clause of the type $a \vee x$ is converted to an inequality $\neg x \le a$, and
- each clause of the type $\neg a \vee x$ is converted into an inequality $a \le x$.

Thus, in the above formula, clauses containing a or $\neg a$ are converted into the following inequalities:

- the clause $a \vee b$ is converted into an inequality $\neg b \le a$;
- the clause $\neg a \vee b$ is converted into an inequality $a \le b$;
- the clause $\neg a \vee \neg b$ is converted into an inequality $a \le \neg b$;
- the clause $a \vee \neg c$ is converted into an inequality $c \le a$; and
- the clause $\neg a \vee \neg c$ is converted into an inequality $a \le \neg c$.

Here, we have:

- two lower bounds for a: $\neg b \le a$ and $c \le a$, and
- three upper bounds for a: $a \le b$, $a \le \neg b$, and $a \le \neg c$.

In other words, we have:

$$\neg b, c \le a \le b, \neg b, \neg c. \tag{1}$$

Each lower bound must be smaller than or equal to each upper bound. So, we get the following inequalities:

- from the first lower bound, we get $\neg b \le b$ and $\neg b \le \neg c$;

- from the second lower bound, we get $c \leq b$, $c \leq \neg b$, and $c \leq \neg c$.

As in the example given in the lecture, the inequality $\neg b \leq b$ is only true when $b = 1$. Similarly, the inequality $c \leq \neg c$ is only satisfied when $c = 0$. For these values b and c, the inequality (1) takes the form

$$0, 0 \leq a \leq 1, 0, 1$$

i.e., the form $0 \leq a \leq 0$. Thus, $a = 0$.

So, the solution is: $a = 0$, $b = 1$, and $c = 0$. In other words, a is false, b is true, and c is false.