Solution to Problem 36

Problem. On the example of the function $f(x) =\sim x$, trace, step by step, how Deutsch-Josza algorithm will conclude that $f(0) \neq f(1)$ while applying f only once.

Solution. The Deutsch-Jozsa algorithm works as follows:

- we start with the state $|0,1\rangle = |0\rangle \otimes |1\rangle$;
- we apply the Hadamard transformation H to both bits;
- then, we apply f;
- after that, we again apply the Hadamard transformation to both bits;
- finally, we measure the first bit of the resulting 2-bit state:
  - if the first bit is 0, we conclude that the function f is constant;
  - if the first bit is 1, we conclude that the function f is not constant.

We start with the state $|0,1\rangle = |0\rangle \otimes |1\rangle$, in which the first bit is in state $|0\rangle$ and the second bit is in state $|1\rangle$. When we apply the Hadamard transformation to both bits, we get

$$H(|0\rangle) \otimes H(|1\rangle) = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).$$

Due to linearity, we get

$$H(|0\rangle) \otimes H(|1\rangle) = \frac{1}{2}|0,0\rangle - \frac{1}{2}|0,1\rangle + \frac{1}{2}|1,0\rangle - \frac{1}{2}|1,1\rangle.$$

Then, we apply the formula $|x,y\rangle \mapsto |y \oplus f(x)\rangle$. In our case, $f(0) = 1$ and $f(1) = 0$, so

$$f(|0,0\rangle) = |0,1\rangle, \quad f(|0,1\rangle) = |0,0\rangle, \quad f(|1,0\rangle) = |1,0\rangle, \quad f(|1,1\rangle) = |1,1\rangle,$$

and, thus,

$$f(H(|0\rangle) \otimes H(|1\rangle)) = \frac{1}{2}|0,1\rangle - \frac{1}{2}|0,0\rangle + \frac{1}{2}|1,0\rangle - \frac{1}{2}|1,1\rangle =$$

$$\frac{1}{2}|0\rangle \otimes |1\rangle - \frac{1}{2}|0\rangle \otimes |0\rangle + \frac{1}{2}|1\rangle \otimes |0\rangle - \frac{1}{2}|1\rangle \otimes |1\rangle.$$

The first two terms have a common factor $|0\rangle$, the third and the fourth one have a common factor $|1\rangle$, so we have

$$f(H(|0\rangle)\otimes H(|1\rangle)) = \frac{1}{\sqrt{2}}|0\rangle\otimes\left(\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle\right)+\frac{1}{\sqrt{2}}|1\rangle\otimes\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).$$

This expression can be equivalently reformulated as

$$f(H(|0\rangle)\otimes H(|1\rangle)) = -\frac{1}{\sqrt{2}}|0\rangle\otimes\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)+\frac{1}{\sqrt{2}}|1\rangle\otimes\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right),$$

and thus, as

$$f(H(|0\rangle)\otimes H(|1\rangle)) = -\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\otimes\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).$$

For both bits, what we have is $H|1\rangle$. Thus, when we apply the Hadamard transformation once again, we get $-|1,1\rangle$.

So, when we measure the first bit, we get 1 with the probability $|-1|^2 = 1$. Thus, the algorithm confirms that $f(0) \neq f(1)$.