Solution to Homework 37

Problem. What class of polynomial hierarchy contains $\Pi_2\mathrm{P}^{\Sigma_2\mathrm{P}}$? Explain your answer.

Solution. For each oracle A, the class $\Pi_2\mathrm{P}^A$ is described by formulas F with 2 quantifiers starting with the universal quantifier in which the main property is feasible with respect to A:

$$F \equiv \forall x_1\, \exists x_2\, C^A(x_1, x_2, x). \tag{1}$$

Here, the fact that $C^{\Sigma_2\mathrm{P}}$ is feasible with respect to the corresponding oracle $A = \Sigma_2\mathrm{P}$ means that this property, in its turn, has the form

$$C^A(x_1, x_2, x) = C^{\Sigma_2\mathrm{P}}(x_1, x_2, x) \equiv$$

$$\exists x_3\, \forall x_4\, C(x_1, x_2, x_3, x_4, x), \tag{2}$$

where the property $C(x_1, x_2, x_3, x_4, x)$ is actually feasible.

Substituting the formula (2) into the expression (1), we get

$$F \equiv \forall x_1\, \exists x_2\, \exists x_3\, \forall x_4\, C(x_1, x_2, x_3, x_4, x). \tag{3}$$

As we know, the two identical neighboring quantifiers can be combined into one, so we have

$$F \equiv \forall x_1\, \exists x_{23}\, \forall x_4\, C(x_1, x_2, x_3, x_4, x). \tag{4}$$

In this expression, we have 3 quantifiers, the first of which is the universal quantifier. Thus, this formula belongs to the class $\Pi_3\mathrm{P}$, i.e.:

$$\Pi_2\mathrm{P}^{\Sigma_2\mathrm{P}} \subseteq \Pi_3\mathrm{P}.$$