

Solution to Problem 2

Problem 2. Prove that if $P(\bar{n})$, $Q(\bar{n})$, $f(\bar{n})$, $g(\bar{n})$, and $h(\bar{n})$ are all primitive recursive, then the function described as

$$F(\bar{n}) = \text{if } (\text{not } P(\bar{n})) \text{ then } f(\bar{n}) \text{ elseif } Q(\bar{n}) \text{ then } g(\bar{n}) \text{ else } h(\bar{n})$$

is also primitive recursive.

Solution. Proof. We want:

- the value $f(\bar{n})$ if $P(\bar{n})$ is false, i.e., if the expression

$$\text{not } P(\bar{n})$$

is true, i.e., equivalently, if $1 \dot{-} P(\bar{n}) = 1$;

- the value $g(\bar{n})$ if $P(\bar{n})$ is true and $Q(\bar{n})$ is true, i.e., if the expression

$$P(\bar{n}) \text{ and } Q(\bar{n})$$

is true, i.e., equivalently, if $P(\bar{n}) \cdot Q(\bar{n}) = 1$;

- the value $h(\bar{n})$ if $P(\bar{n})$ is true and $Q(\bar{n})$ is false, i.e., if the expression

$$P(\bar{n}) \text{ and not } Q(\bar{n})$$

is true, i.e., equivalently, if $P(\bar{n}) \cdot (1 \dot{-} Q(\bar{n})) = 1$.

Thus, it makes sense to take

$$F(\bar{n}) = (1 \dot{-} P(\bar{n})) \cdot f(\bar{n}) + P(\bar{n}) \cdot Q(\bar{n}) \cdot g(\bar{n}) + P(\bar{n}) \cdot (1 \dot{-} Q(\bar{n})) \cdot h(\bar{n}).$$

Then:

- if $P(\bar{n})$ is false, i.e., if $P(\bar{n}) = 0$, then $1 \dot{-} P(\bar{n}) = 1$, so

$$F(\bar{n}) = 1 \cdot f(\bar{n}) + 0 \cdot g(\bar{n}) + 0 \cdot h(\bar{n}) = f(\bar{n});$$

- if $P(\bar{n})$ is true and $Q(\bar{n})$ is true, then $1 \dot{-} P(\bar{n}) = 1 \dot{-} Q(\bar{n}) = 0$ and $P(\bar{n}) \cdot Q(\bar{n}) = 1$, so

$$F(\bar{n}) = 0 \cdot f(\bar{n}) + 1 \cdot g(\bar{n}) + 0 \cdot h(\bar{n}) = g(\bar{n});$$

- if $P(\bar{n})$ is true and $Q(\bar{n})$ is false, then $1 \dot{-} P(\bar{n}) = Q(\bar{n}) = 0$ and $P(\bar{n}) \cdot (1 \dot{-} Q(\bar{n})) = 1$, so

$$F(\bar{n}) = 0 \cdot f(\bar{n}) + 0 \cdot g(\bar{n}) + 1 \cdot h(\bar{n}) = h(\bar{n}).$$