

## Solution to Problem 23

**Problem.** Let us consider cases when the set  $A$  is decidable and the complements to the sets  $B$  and  $C$  are r.e. Give four examples of such cases:

- an example when the union  $A \cup B \cup C$  of the three sets  $A$ ,  $B$ , and  $C$  is decidable,
- an example when the union  $A \cup B \cup C$  of the three sets  $A$ ,  $B$ , and  $C$  is not decidable,
- an example when the intersection of the three sets  $A$ ,  $B$ , and  $C$  is decidable, and
- an example when the intersection of the three sets  $A$ ,  $B$ , and  $C$  is not decidable.

**Solution.** Finding decidable examples is easy: it is sufficient to have decidable sets  $A$ ,  $B$ , and  $C$ . For example:

- if  $A = B = C = \emptyset$ , then their union  $A \cup B \cup C$  is the empty set and thus, decidable;
- if  $A = B = C = \emptyset$ , then their intersection  $A \cap B \cap C$  is the empty set and thus, decidable.

Non-decidable examples are not so easy. To find such examples, let us recall that the union and intersection of decidable sets is decidable. Since the set  $A$  is decidable, we need to find the set  $B$  or  $C$  which is not decidable. The only set for which we know that its complement is not decidable (it is even not r.e.) is the halting set. So, as  $C$  let us take the complement  $-H$  to the halting set. Here is one of the possible ways to find such examples:

- if  $A = B = \emptyset$  and  $C = -H$ , then their union  $A \cup B \cup C$  is equal to  $-H$  and is, thus, not decidable;
- if  $A = B = N$  (the set of all natural numbers) and  $C = -H$ , then their intersection  $A \cap B \cap C$  is equal to  $-H$  and is, thus, not decidable.