

## Solution to Homework 38

**Problem.** What class of polynomial hierarchy contains  $\Sigma_2\text{P}^{\Pi_2\text{P}}$ ? Explain your answer.

**Solution.** For each oracle  $A$ , the class  $\Sigma_2\text{P}^A$  is described by formulas  $F$  with 2 quantifiers starting with the existential quantifier in which the main property is feasible with respect to  $A$ :

$$F \equiv \exists x_1 \forall x_2 C^A(x_1, x_2, x). \quad (1)$$

Here, the fact that  $C^{\Pi_2\text{P}}$  is feasible with respect to the corresponding oracle  $A = \Pi_2\text{P}$  means that this property, in its turn, has the form

$$\begin{aligned} C^A(x_1, x_2, x) &= C^{\Pi_2\text{P}}(x_1, x_2, x) \equiv \\ &\forall x_3 \exists x_4 C(x_1, x_2, x_3, x_4, x), \end{aligned} \quad (2)$$

where the property  $C(x_1, x_2, x_3, x_4, x)$  is actually feasible.

Substituting the formula (2) into the expression (1), we get

$$F \equiv \exists x_1 \forall x_2 \forall x_3 \exists x_4 C(x_1, x_2, x_3, x_4, x). \quad (3)$$

As we know, the two identical neighboring quantifiers can be combined into one, so we have

$$F \equiv \exists x_1 \forall x_{23} \exists x_4 C(x_1, x_2, x_3, x_4, x). \quad (4)$$

In this expression, we have 3 quantifiers, the first of which is the existential quantifier. Thus, this formula belongs to the class  $\Sigma_3\text{P}$ , i.e.:

$$\Sigma_2\text{P}^{\Pi_2\text{P}} \subseteq \Sigma_3\text{P}.$$