From Interval Computations to Processing Fuzzy Data: Main Ideas

Slides for Interval Computations class

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1. Classification of Practical Problems

1) We want to *learn* what is happening in the world.

- In particular, we want to know the numerical values of different quantities:
  - distances,
  - masses,
  - charges,
  - coordinates, etc.

2) Based on these values, we would like to *predict* how the state of the world will change over time.

3) Finally, we would like to find out what *changes* we need to make to get the desired results.

- A real-life problem often involves solving subproblems of all three types.

2. Classification of Practical Problems (cont-d)

- Learning the current state of the world and predicting the future are usually classified as *science*.
- The tasks of finding the appropriate change are usually classified as *engineering*.
- Measuring the flow of Rio Grande and predicting how this flow will change over time are problems of science.
- Finding the best way to change this flow (e.g., by building a levee) is a problem of engineering.

3. First Class of Practical Problems: Learning the State of the World

- Let us start with the first class of practical problems: the problem of learning the state of the world.
- In particular, that we want to know the numerical values of different quantities y that characterize this state.
- Some quantities y we can simply directly measure, e.g.:
  - when we want to know the current state of a patient in a hospital,
  - we can measure the patient's body temperature, blood pressure, weight, and many other important characteristics.
- In some situations, we do not even need to measure:
  - we can simply ask an expert,
  - and the expert will provide us with an (approximate) value $\widetilde{y}$ of the quantity y.

4. Sometimes, We Cannot Directly Measure

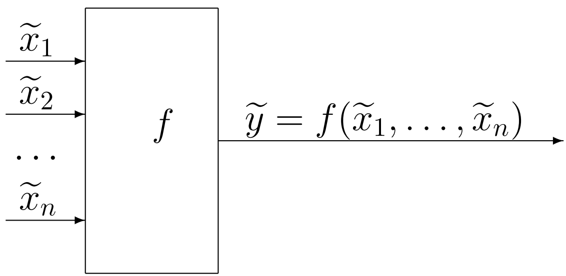
- However, many other quantities of interest are difficult or even important to measure or estimate directly.
- Examples of such quantities include the amount of oil in a given well or a distance to a star.
- Since we cannot directly measure these quantities, the only way to get information about them is:
  - to measure (or ask an expert to estimate) some other easier-to-measure quantities $x_1, \ldots, x_n$,
  - then to estimate y based on the measured values $\widetilde{x}_i$ of these auxiliary quantities x_i.
- For example, to estimate the amount of oil in a given well, we perform *seismic* experiments:
  - we set up small explosions at some locations and
  - measure the resulting seismic waves at different distances from the location of the explosion.

5. Sometimes, We Can't Measure (cont-d)

- To find the distance to a faraway star, we measure:
  - the direction to the star from different locations on Earth (and/or in different seasons) and
  - the coordinates of (and the distances between) the locations of the corresponding telescopes.
- To estimate the value of the desired quantity y, we must know the relation:
  - between y and
  - the easier-to-measure (or easier-to-estimate) quantities $x_1, \ldots, x_n$.
- Specifically, we want to use the estimates of x_i to come up with an estimate for y.

6. Need for an Algorithm

- Thus, the relation between y and x_i must be given in the form of an *algorithm* $f(x_1, \ldots, x_n)$ which:
  - transforms the values of x_i
  - into an estimate for y.
- Once we know this algorithm f and the measured values $\widetilde{x}_i$ of the auxiliary quantities, we can estimate y:



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7. Algorithms Can Have Different Complexity

- In different practical situations, we have algorithms f of different complexity.
- To find the distance to star, we can usually have an explicit trigonometric formula.
- In this case, f is a simple formula.
- To find the amount of oil, we must numerically solve a complex partial differential equation.
- In this case, f is a complex iterative algorithm for solving this equation.
- When the values x_i are obtained by measurement, this 2-stage process does involve measurement.
- To distinguish it from *direct* measurements, this 2-stage process is called an *indirect* measurement.

8. Possible Exam Questions

- What is indirect measurement?
- Why do we need indirect measurements? Provide examples.

9. Second Class of Practical Problems: Predicting the Future State of the World

- Once
  - we know the values of the quantities $y_1, \ldots, y_m$ characterizing the current state of the world,
  - we can start predicting the future state of the world,
  - i.e., the future values of these quantities.
- To be able to predict the future value z of each of these quantities, we must know:
  - how exactly this value z
  - depends on the current values $y_1, \ldots, y_m$.
- Specifically, we want to use the known estimates $\widetilde{y}_i$ for y_i to come up with an estimate for z.

10. Need for an Algorithm

- Thus, the relation between z and y_i must be given in the form of an *algorithm* $g(y_1, \ldots, y_m)$ which:
  - transforms the values of y_i
  - into an estimate for z.
- Once:
  - we know this algorithm g and the estimates $\widetilde{y}_i$ for the current values of the quantities,
  - we can estimate z as $\widetilde{z} = g(\widetilde{y}_1, \ldots, \widetilde{y}_n)$.
- The corresponding algorithm g can be very complicated and time-consuming.
- This is, e.g., how weather is predicted now:
  - weather prediction requires so many computations
  - that it can only be performed on fast supercomputers.

11. The General Notion of Data and Knowledge Processing

- So far, we have analyzed two different classes of practical problems:
  - the problem of *learning* the current state of the world (i.e., the problem of indirect measurement),
  - and the problem of *predicting* the future state of the world.
- From the *practical* viewpoint, these two problems are drastically different.
- However, as we have seen, from the *computational* viewpoint, these two problems are very similar.

12. Data and Knowledge Processing (cont-d)

- In both problems:
  - we start with the estimates $\widetilde{x}_1, \ldots, \widetilde{x}_n$ for the quantities $x_1, \ldots, x_n$, and then
  - we apply the known algorithm f to these estimates,
  - this results in an estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ for the desired quantity y.
- When the inputs come from measurements, they constitute *data*.
- The computational part of the corresponding procedure is called *data processing*.
- When the inputs come from experts, it constitutes *knowledge*.
- The computational part of the corresponding procedure is called *knowledge processing*.

13. Possible Exam Questions

- What is data processing? Give examples.
- What is knowledge processing? Give examples.
- Why do we need data processing and knowledge processing?
- What is the difference between data processing and knowledge processing?

14. Third Class of Practical Problems: How to Change the World

- Once:
  - we know the current state of the world and
  - we know how to predict the consequences of different decisions (designs, etc.),
  - it is desirable to find the decision (design, etc.) which guarantees the given results.
- Depending on what we want from this design:
  - we can subdivide all the problems from this class
  - into two subclasses.
- In both subclasses, the design must satisfy some constraints.
- Thus, we are interested in finding a design that satisfies all these constraints.

15. Two Subclasses

- In some practical situations, satisfaction of all these constraints is all we want.
- In general, there may be several possible designs which satisfy given constraints.
- In the problems from the first subclass, we do not have any preferences for one of these designs.
- Any one of them will suffice.
- Such problems are called the problems of *constraint satisfaction*.

16. Two Subclasses (cont-d)

- In other practical situations, we do have a clear preference between different designs x.
- This preference is usually described in terms of an *objective function* $F(x)$:
  - more preferable designs x
  - correspond to larger values of $F(x)$.
- In such situation:
  - among all the designs which satisfy given constraints,
  - we would like to find a design x for which the value $F(x)$ of the given objective function is the largest.
- Such problems are called *optimization problems*.

17. Possible Exam Questions

- What is constraint satisfaction? Why do we need it? Provide examples.
- What is optimization? Why do we need it? Provide examples.
- What is the difference between constraint satisfaction and optimization?

18. Need to Process Fuzzy Uncertainty

- In many practical situations, we only have expert estimates for the inputs x_i.
- Sometimes, experts provide guaranteed bounds on the values of x_i.
- Sometimes, they even provide the probabilities of different values within these bounds.
- However, such cases are rare.
- Usually, the experts' opinions are described by imprecise ("fuzzy") words from natural language.
- For example, an expert can say that:
  - the value x_i of the i-th quantity is approximately equal to 1.0,
  - with an accuracy most probably of about 0.1.

19. Need to Process Fuzzy Uncertainty (cont-d)

- Based on such "fuzzy" information, what can we say about $y = f(x_1, \ldots, x_n)$?
- The need to process such "fuzzy" information was first emphasized in the early 1960s by Lotfi Zadeh.
- He designed a special technique for such processing called *fuzzy logic*.

20. Possible Exam Question

- What is fuzzy processing? Why is it called "fuzzy"?
- Why do we need fuzzy processing? Give an example.
- Who was the first to develop fuzzy techniques?

21. Processing Fuzzy Uncertainty: Main Idea

- A value y is a reasonable value of the desired quantity if $y = f(x_1, \ldots, x_n)$ for some reasonable values x_i.
- In other words: if for some values $x_1, \ldots, x_n$, x_1 is reasonable, x_2 is reasonable, …, and $y = f(x_1 \ldots, x_n)$.
- Thus, to describe to what extent different values of y are reasonable, we must be able:
  - to describe to what extent (to what degree) different values of x_i are reasonable, and
  - to combine these degrees into the desired degree of belief in reasonability of y.

22. Degrees of Belief

- Let us first introduce the basic concept of degrees of belief.
- For example, we would like to estimate to what extent the value $x_i = 0.89$ is consistent with the statement:
  - *the value x_i of the i-th quantity is approximately equal to 1.0,*
  - *with an accuracy most probably about 0.1.*
- In the absence of uncertainty, every statement is either true or false.
- In the computer, "true" is usually represented as 1, and "false" as 0.

23. Degrees of Belief (cont-d)

- It is thus reasonable to use numbers between 0 and 1 to represent levels of confidence intermediate between:
  - the absolute confidence that a given statement is true and
  - the absolute confidence that a given statement is false.
- How do we determine this degree of confidence?
- For example, we can:
  - ask several (N) experts whether $x_i = 0.89$ is consistent with the above statement, and
  - if M of them reply "yes", take the ratio M/N as the desired degree of confidence.

24. Degrees of Belief (cont-d)

- If we do not have access to numerous experts, we can simply:
  - ask the only available expert to describe his or her degree of confidence
  - by marking a number on a scale from 0 to N (e.g., on a scale from 0 to 5).
- If an expert marks his or her degree as M, we take the ratio M/N as the desired degree of confidence.

25. Possible Exam Questions

- Why degree of belief 1 usually corresponds to absolute confidence, and 0 to absolute lack of confidence?
- Describe two ways to elicit degrees of belief from experts.
- Suppose that we are trying to formalize a medical rule about small tumors.
- Out of 20 experts, 16 think that 0.3 mm is small.
- What will be the corresponding degree of belief that 0.3 mm is small?
- An expert marks her degree of confidence that 0.3 mm is small as 3.5 on a 0 to 5 scale.
- What is the resulting degree of belief?

26. Membership Functions

- To formally describe the original expert's statement S about x_i, we need to know:
  - for every real number x_i,
  - the degree $\mu_S(x_i)$ to which this real number is consistent with this statement S.
- In practice, we can only ask finitely many questions.
- So, no matter how many questions we ask, we can only find $\mu_S(x_i)$ for finitely many values x_i.
- To estimate the values $\mu_S(x_i)$ for all other real numbers x_i, we must use interpolation and extrapolation.
- Usually, a piece-wise linear interpolation is used.
- Sometimes, a more sophisticated procedure is applied: e.g., a piecewise quadratic interpolation.

27. Membership Functions (cont-d)

- The function $\mu_S(x_i)$ which is obtained by this interpolation is called a *membership function*.
- This function describes:
  - for every real number x_i,
  - the degree $\mu_S(x_i)$ to which this real number is consistent with this statement S.

28. Possible Exam Questions

- What is a membership function?
- How can we determine the membership function?
- We know that $\mu(0) = 1$ and $\mu(2) = 0$. Use linear interpolation to find $\mu(x)$ for $x \in [0, 2]$.
- *Reminder:* if we know that $f(x_1) = y_1$ and $f(x_2) = y_2$, then as an estimate for $f(x)$, we take

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1).$$

- In our case, $x_1 = 0$, $x_2 = 2$, $y_1 = 1$, $y_2 = 0$, so

$$y = 1 + \frac{0 - 1}{2 - 0} \cdot (x - 0) = 1 - x/2.$$

- *Now students have to work:* we know that $\mu(-2) = 0$, $\mu(0) = 1$, use linear interpolation to find $\mu(-1.6)$.

29. Need for "And"- and "Or"-Operations: t-Norms and t-Conorms

- As we have mentioned earlier:
  - we are not directly interested
  - in the degree to which a given real number x_i is consistent with the expert's knowledge S_i about the i-th input.
- We are mainly interested in the degree to which:
  - the value x_1 is consistent with the knowledge about the first input *and*
  - the value x_2 is consistent with the knowledge about the second input *and*
  - … *and*
  - the value x_n is consistent with the knowledge about the n-th input.

30. "And"- and "Or"-Operations (cont-d)

- In principle, we can determine the degree of belief in such a composite statement by asking an expert:
  - for each possible combination of values $x_1, x_2, \ldots, x_n$,
  - what is the degree to which this combination is consistent with all the available expert knowledge.
- However, as we have mentioned earlier:
  - even for a single input,
  - we cannot realistically elicit degrees of confidence about too many values.
- If we consider N possible values of each input, then:
  - we would need to elicit the expert's degree of confidence
  - about $N^n \gg N$ possible combinations.

31. “And”- and “Or”-Operations (cont-d)

- Eliciting N^n values is even less realistic than eliciting N values.
- We cannot directly elicit the expert's degree of confidence in all composite statements.
- So, a natural idea is:
  - to estimate the degree of confidence in the composite statement
  - based on the degrees of confidence in individual statements
  - such as “x_i is consistent with the expert's knowledge S_i about the i-th input.”
- How can we come up with such an estimate?

32. "And"- and "Or"-Operations (cont-d)

- Let us reformulate this estimation problem:
  - we know the expert's degree of confidence in statements A_1, A_2, ..., A_n, and
  - we want to estimate the expert's degree of confidence in a composite statement

$$A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n,$$

  - i.e., "A_1 and A_2 and ... and A_n".
- Here, e.g., $A_1 \,\&\, A_2 \,\&\, A_3$ can be represented as $(A_1 \,\&\, A_2) \,\&\, A_3$.
- So, it is sufficient to solve this estimation problem for the case of two statements.

33. "And"- and "Or"-Operations (cont-d)

- Once we have a solution for this particular case, we will then be able to solve the general problem as well:
  - first, we apply the two-statement solution to the degrees of certainty in A_1 and A_2,
  - we thus get an estimate for the expert's degree of certainty in $A_1 \,\&\, A_2$;
  - then, we apply the same solution to the degrees of certainty in $A_1 \,\&\, A_2$ and A_3,
  - we thus get an estimate for the expert's degree of certainty in $A_1 \,\&\, A_2 \,\&\, A_3$;
  - after that, we apply the same solution to the degrees of certainty in $A_1 \,\&\, A_2 \,\&\, A_3$ and A_4,
  - we thus get an estimate for the expert's degree of certainty in $A_1 \,\&\, A_2 \,\&\, A_3 \,\&\, A_4$;
  - etc.

34. "And"- and "Or"-Operations (cont-d)

- Eventually, we will get the degree of confidence in the desired composite statement $A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n$.
- Thus, we need a procedure that would transform:
  - the degree of belief d_1 in a statement A_1 and
  - the degree of belief d_2 in a statement A_2
  - into a (reasonable) estimate for a degree of belief in a composite statement $A_1 \,\&\, A_2$.
- Let us denote the estimate corresponding to given values d_1 and d_2 by $f_\&(d_1, d_2)$.
- The procedure $f_\&$ is called an *"and"-operation*, or, for historical reasons, a *t-norm*.
- This procedure maps:
  - degrees of belief d_1 and d_2 in statements A_1 and A_2
  - into a degree of belief $d = f_\&(d_1, d_2)$ in $A_1 \,\&\, A_2$.

35. "Or"-Operations

- Similarly:
  - to estimate the degree of belief in a composite statement $A_1 \vee A_2$ ("A_1 or A_2"),
  - we need a procedure $f_\vee$ that maps:
    * degrees of belief d_1 and d_2 in statements A_1, A_2
    * into a degree of belief $d = f_\vee(d_1, d_2)$ in $A_1 \vee A_2$.
- Such a procedure is called an *"or"-operation*.
- In logic:
  - "or" is a kind of dual to "and", and
  - "or"-operation can be viewed as a dual to an "and"-operation (t-norm).
- Because of this duality, an "or"-operation is also called a *t-conorm*.

36. Possible Exam Questions

- What is an "and"-operation?
- Why do we need "and"-operations? Why cannot we explicitly elicit all the degrees from the experts?
- What is an "or"-operation?
- Why do we need "or"-operations?

37. Properties of "And"- and "Or"-Operations

- From the intended meaning of the "and"- and "or"-operations, we can deduce reasonable properties.
- For example, intuitively, "A_1 and A_2" means the same as "A_2 and A_1".
- Thus, it is reasonable to require that:
  - our estimate $f_\&(d_1, d_2)$ for the degree of confidence in "A_1 and A_2"
  - should be the same our estimate $f_\&(d_2, d_1)$ for the degree of confidence in "A_2 and A_1".
- In other words, we must have $f_\&(d_1, d_2) = f_\&(d_2, d_1)$ for all possible values of d_1 and d_2.
- In mathematical terms, this means that the function $f_\&$ must be *commutative*.

38. Associativity

- Similarly, "(A_1 and A_2) and A_3" means the same as "A_1 and (A_2 and A_3)".
- Indeed, both mean the same as "A_1 and A_2 and A_3".
- For each "and"-operation $f_\&$, the expression "(A_1 and A_2) and A_3" means that we:
  - first estimate the degree of belief in "A_1 and A_2" as $f_\&(d_1, d_2)$, and
  - then estimate the degree of belief in "(A_1 and A_2) and A_3" as $f_\&(f_\&(d_1, d_2), d_3)$.
- Similarly, the expression "A_1 and (A_2 and A_3)" means that we:
  - first estimate the degree of belief in "A_2 and A_3" as $f_\&(d_2, d_3)$, and
  - then estimate the degree of belief in "A_1 and (A_2 and A_3)" as $f_\&(d_1, f_\&(d_2, d_3))$.

39. Associativity and Other Properties

- Since the expressions are equivalent, it is reasonable to require that these estimates coincide.
- So, we must have $f_\&(f_\&(d_1, d_2), d_3) = f_\&(d_1, f_\&(d_2, d_3))$ for all possible values of d_1, d_2, and d_3.
- In mathematical terms, this means that the function $f_\&$ must be *associative*.
- There are several other reasonable properties of "and"-operations.
- For example, "A_1 and A_2" implies A_1.
- So, our degree of belief in the composite statement "A_1 and A_2" cannot exceed our degree of belief in A_1.

40. Other Properties (cont-d)

- Thus, it is reasonable to require that:
  - the estimate $f_\&(d_1, d_2)$ for this degree of belief
  - should also not exceed our degree of belief d_1 in the statement A_1.
- In other words, we should have $f_\&(d_1, d_2) \le d_1$ for all possible values of d_1 and d_2.
- If A_1 is absolutely true (i.e., $d_1 = 1$), then intuitively:
  - the composite statement "A_1 and A_2"
  - has exactly the same truth value as A_2.
- Thus, it is reasonable to require that $f_\&(1, d_2) = d_2$ for all possible values of d_2.

41. Other Properties (cont-d)

- On the other hand:
  - if A_1 is absolutely false (i.e., $d_1 = 0$),
  - then the composite statement "A_1 and A_2" should also be absolutely false,
  - no matter how much we may believe in A_2.
- Thus, it is reasonable to require that $f_\&(0, d_2) = 0$ for all possible values of d_2.
- Finally:
  - if, due to a new evidence, our degree of belief in one of the statements A_1 and A_2 increases,
  - the resulting degree of belief in "A_1 and A_2" will either increase or stay the same,
  - but it cannot decrease.

42. Other Properties (cont-d)

- Thus, it is it is reasonable to require that the operation $f_\&$ be *monotonic* in the sense that:
  - if $d_1 \le d'_1$ and $d_2 \le d'_2$,
  - then $f_\&(d_1, d_2) \le f_\&(d'_1, d'_2)$.
- All these properties are indeed required of an "and"-operation (t-norm).
- Similarly, it is reasonable to require that an "or"-operation (t-conorm) $f_\vee$ should be:
  - commutative,
  - associative,
  - monotonic, and
  - satisfy the conditions $d_1 \le f_\vee(d_1, d_2)$, $f_\vee(1, d_2) = 1$, and $f_\vee(0, d_2) = d_2$ for all d_1 and d_2.

43. Possible Exam Questions

- Why should an "and"-operation be commutative? Explain.
- Why should an "and"-operation be associative? Explain.
- Why should an "or"-operation be commutative? Explain.
- Why should an "or"-operation be associative? Explain.

44. Simplest "And"- and "Or"-Operations: Derivation

- There exist many different "and"- and "or"-operations which satisfy the above properties.
- In some applications such as fuzzy control, it is crucial to select appropriate operations:
  - we can use the additional degrees of freedom
  - to tune the resulting control and
  - thus make it an even better fit for the corresponding objective function.
- However, in knowledge processing:
  - when we are very uncertain about the inputs,
  - it is probably more reasonable to select the simplest "and"- and "or"-operations
  - which are consistent with the expert knowledge.

45. Simplest "And"-Operation (cont-d)

- To select such operations, it makes sense to consider yet another property of "and" and "or" – that:
  - for every statement A,
  - "A and A" means the same as simply A.
- Thus, it is reasonable to require that:
  - for every statement A with a degree of confidence d,
  - our estimate $f_\&(d, d)$ of the expert's degree of confidence in "A and A"
  - should be the same as the original degree of confidence d in the original statement A.
- Thus, it is reasonable to require that $f_\&(d, d) = d$ for all possible values of d.
- In mathematical terms, this means that the function $f_\&$ must be *idempotent*.

46. Simplest "Or"-Operation

- Similarly, "A or A" means the same as simply A.
- So, it is reasonable to require that $f_\vee(d, d) = d$ for all possible values of d.
- So, the function $f_\vee$ must also be *idempotent*.

47. Uniqueness of Idempotent "And"-Operation

- Let us first show that the only idempotent "and"-operation is $f_\&(d_1, d_2) = \min(d_1, d_2)$.
- Without loss of generality, let us assume that $d_1 \le d_2$.
- In this case, the desired equality takes the form

$$f_\&(d_1, d_2) = d_1.$$

- Since the operation $f_\&$ is idempotent, we have

$$f_\&(d_1, d_1) = d_1.$$

- Due to $d_1 \le d_2$, monotonicity implies that $f_\&(d_1, d_1) \le f_\&(d_1, d_2)$, hence $d_1 \le f_\&(d_1, d_2)$.
- On the other hand, for an "and"-operation, we always have $f_\&(d_1, d_2) \le d_1$.
- So, we can conclude that $f_\&(d_1, d_2) = d_1$, i.e., indeed, $f_\&(d_1, d_2) = \min(d_1, d_2)$.

48. Uniqueness of Idempotent "Or"-Operation

- Let us now prove that the only idempotent "or"-operation is $f_\vee(d_1, d_2) = \max(d_1, d_2)$.
- Without loss of generality, let us again assume that
$$d_1 \le d_2.$$
- In this case, the desired equality takes the form
$$f_\vee(d_1, d_2) = d_2.$$
- Since the operation $f_\vee$ is idempotent, we have
$$f_\vee(d_2, d_2) = d_2.$$
- Due to $d_1 \le d_2$, monotonicity implies that $f_\vee(d_1, d_2) \le f_\vee(d_2, d_2)$, hence $f_\vee(d_1, d_2) \le d_2$.
- On the other hand, for an "or"-operation, we always have $d_2 \le f_\vee(d_1, d_2)$, so $f_\vee(d_1, d_2) = d_2$, i.e.,
$$f_\vee(d_1, d_2) = \max(d_1, d_2).$$

49. Simplest "And"- and "Or"–Operations: General

- $f_\&(d_1, d_2) = \min(d_1, d_2)$ and $f_\vee(d_1, d_2) = \max(d_1, d_2)$ were actually the first designed by L. Zadeh.
- They are still actively used in various applications of fuzzy techniques.
- If the expert's degree of confidence in A_1 is 0.6 and in A_2 is 0.6, then:
  - what is the expert's degree of confidence in $A \,\&\, B$?
  - what is the expert's degree of confidence in $A \vee B$?

50. Possible Exam Questions

- Why should an "and"-operation be idempotent? Explain.
- Why should an "or"-operation be idempotent? Explain.
- What is the only idempotent "and"-operation? Who invented it? Is it used in practice?
- What is the only idempotent "or"-operation? Who invented it? Is it used in practice?

51. Fuzzy Data Processing

- Let us apply the above simple operations to knowledge processing, i.e., to processing fuzzy uncertainty.
- In this situation, we know an algorithm $y = f(x_1, \ldots, x_n)$.
- This algorithm relates:
  - the value of the desired difficult-to-estimate quantity y
  - with the values of easier-to-estimate auxiliary quantities $x_1, \ldots, x_n$.
- We also have expert knowledge about each of the quantities x_i.
- For each i, this knowledge is described in terms of the corresponding membership function $\mu_i(x_i)$.

52. Fuzzy Data Processing (cont-d)

- For each i and for each value x_i, the value $\mu_i(x_i)$ is:
  - the degree of confidence
  - that this value is indeed a possible value of the i-th quantity.
- Based on this information, we want to find the membership function $\mu(y)$ which describes:
  - for each real number y,
  - the degree of confidence that this number is a possible value of the desired quantity.

53. Fuzzy Data Processing (cont-d)

- As we have mentioned earlier, y is a possible value of the desired quantity if:
  - for some values $x_1, \ldots, x_n$,
  - x_1 is a possible value of the first input quantity, and
  - x_2 is a possible value of the 2nd input quantity,
  - $\ldots$, and
  - we have $y = f(x_1 \ldots, x_n)$.
- We know:
  - that the degree of confidence that x_1 is a possible value of the first input quantity is equal to $\mu_1(x_1)$,
  - that the degree of confidence that x_2 is a possible value of the 2nd input quantity is equal to $\mu_2(x_2)$,
  - etc.

54. Fuzzy Data Processing (cont-d)

- The degree of confidence $d(y, x_1, \ldots, x_n)$ in an equality $y = f(x_1 \ldots, x_n)$ is, of course, equal:
  - to 1 if this equality holds, and
  - to 0 if this equality does not hold.
- We have already agreed to represent "and" as min.
- Thus, for each combination of values $x_1, \ldots, x_n$,
  - the degree of confidence in a composite statement

    "x_1 is a possible value of the first input quantity, and x_2 is a possible value of the second input quantity, ..., and $y = f(x_1 \ldots, x_n)$"

  - is equal to

$$\mu(y) = \min(\mu_1(x_1), \mu_2(x_2), \ldots, d(y, x_1, \ldots, x_n)).$$

55. Fuzzy Data Processing (cont-d)

- We got the expression

$$\mu(y) = \min(\mu_1(x_1), \mu_2(x_2), \ldots, d(y, x_1, \ldots, x_n)).$$

- We can simplify this expression if we consider two possible cases:
  - when the equality $y = f(x_1 \ldots, x_n)$ holds, and
  - when this equality does not hold.
- When the equality $y = f(x_1 \ldots, x_n)$ holds, we get $d(y, x_1, \ldots, x_n) = 1$.
- Thus, the above degree of confidence is equal to

$$\min(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)).$$

- When the equality $y = f(x_1 \ldots, x_n)$ does not hold, we get $d(y, x_1, \ldots, x_n) = 0$.

56. Fuzzy Data Processing (cont-d)

- Thus, the above degree of confidence is equal to 0.
- We want to combine these degrees of belief into a single degree of confidence that

  "for some values $x_1, \ldots, x_n$, x_1 is a possible value of the first input quantity, and x_2 is a possible value of the first input quantity, ..., and $y = f(x_1 \ldots, x_n)$".

- The words "for some values $x_1, \ldots, x_n$" means that the following composite property holds:
  - either for one combination of real numbers $x_1, \ldots, x_n$,
  - or for another combination
  - until we exhaust all (infinitely many) such combinations.

57. Fuzzy Data Processing (cont-d)

- We have already agreed to represent "or" as max.
- Thus, the desired degree of confidence $\mu(y)$ is equal:
  - to the maximum of the degrees
  - corresponding to different combinations $x_1, \ldots, x_n$.
- Since we have infinitely many possible combinations, the maximum is not necessarily attained.
- So we should, in general, consider supremum instead of maximum:

$$\mu(y) = \sup \min(\mu_1(x_1), \mu_2(x_2), \ldots, d(y, x_1, \ldots, x_n)).$$

- Here, the supremum is taken over all possible combinations.
- We know that the maximized degree is non-zero only when $y = f(x_1 \ldots, x_n)$.

58. Zadeh's Extension Principle

- So, it is sufficient to only take supremum over such combinations.
- For such combinations, we can omit the term $d(y, x_1, \ldots, x_n)$ in the maximized expression.
- So, we arrive at the following formula for $\mu(y)$:

$$\sup\{\min(\mu_1(x_1), \ldots, \mu_n(x_n)) : y = f(x_1, \ldots, x_n)\}.$$

- This formula describes a reasonable way to extend an arbitrary data processing algorithm $f(x_1, \ldots, x_n)$:
  - from real-valued inputs
  - to a more general case of fuzzy inputs.
- It was first proposed by L. Zadeh and is thus called *Zadeh's extension principle.*
- This is the main formula that describes knowledge processing under fuzzy uncertainty.

59. Fuzzy Data Processing Can Be Reduced to Interval Computations

- We will show that from the computational viewpoint:
  - the application of this formula
  - can be reduced to interval computations.
- This reduction is how knowledge processing under fuzzy uncertainty is usually done.

60. Possible Exam Questions

- Describe the main problem of fuzzy data processing in precise terms.
- Formulate Zadeh's extension principle.
- *For extra credit:* explain how Zadeh's extension principle is derived.

61. An Alternative Set Representation of a Membership Function: Alpha-Cuts

- In some situations, an expert knows exactly which values of x_i are possible and which are not.
- In this situation, the expert's knowledge can be naturally represented by the *set* of all possible values.
- In general, the expert's knowledge is fuzzy:
  - we may still have some values about which the expert 100% believes that they are possible, and
  - we may still have some values about which the expert 100% believes that they are impossible, but
  - in general, the expert is not 100% confident about which values of x_i are possible and which are not.

62. Alpha-Cuts (cont-d)

- For example, a geophysicist may be confident:
  - that the density x_i of some mineral can take on values ranging from 3.4 to 3.7 g/cm^3, and
  - she may know that values smaller than 3.0 or larger than 4.0 are absolutely impossible, but
  - she is not sure whether values from 3.0 to 3.4 or from 3.7 to 4.0 are indeed realistically possible.
- As we have mentioned, the ultimate purpose of the measurements and estimates is to make decisions.

63. Alpha-Cuts (cont-d)

- In the geophysical example, we have measured the density at a certain depth, and we need to decide:
  - whether it is possible that we have the desired mineral
  - in this case we should undertake more measurements, or
  - whether it is not possible that we have the desired mineral
  - in this case we should not waste our resources on this region and move to more promising regions.
- In practice, decisions are made under uncertainty:
  - if we only have a fuzzy expert description of possible values, as a membership function $\mu_S(x_i)$,
  - which values x_i should we then classify as possible ones and which as impossible?

64. Alpha-Cuts (cont-d)

- Under uncertainty, a reasonable idea is to select a threshold $\alpha \in (0, 1]$.
- In this case,
  - all the values x_i for which the expert's degree of confidence is strong enough ($\mu_S(x_i) \geq \alpha$)
  - are classified as possible.
- Similarly:
  - all the values x_i for which the expert's degree of confidence is not sufficiently strong ($\mu_S(x_i) < \alpha$)
  - are classified as impossible.
- The resulting set of possible elements is called the α-*cut* of the membership function $\mu_S(x_i)$:

$$\mathbf{x}_i(\alpha) \stackrel{\text{def}}{=} \{x_i : \mu_S(x_i) \geq \alpha\}$$

65. How to Select α

- The choice of a threshold α depends on the practical problem.
- For example:
  - if we are looking for a potentially very valuable mineral deposit,
  - then it makes sense to continue prospecting even when our degree of confidence is not very high.
- In this case, it makes sense to select a reasonably small threshold α.
- On the other hand:
  - if the potential benefit is not high and our resources are limited,
  - it makes sense to limit our search to highly promising regions, i.e., to select a high threshold α.

66. Alpha-Cuts (cont-d)

- To adequately describe the expert knowledge irrespective of an application, we therefore need to know:
  - the α-cuts
  - corresponding to different thresholds α.
- Each α-cut $\mathbf{x}_i(\alpha)$ describes the set of values which are possible with degree of confidence at least α.
- By definition, α-cuts corresponding to different α are *nested*:
  - when $\alpha \leq \alpha'$,
  - then $\mu_S(x_i) \geq \alpha'$ implies $\mu_S(x_i) \geq \alpha$ and thus,

$$\mathbf{x}_i(\alpha') = \{x_i : \mu_S(x_i) \geq \alpha'\} \subseteq \mathbf{x}_i(\alpha) = \{x_i : \mu_S(x_i) \geq \alpha\}.$$

67. From Alpha-Cuts to Membership Functions

- It is worth mentioning that:
  - if we know the α-cuts $\mathbf{x}_i(\alpha) = \{x_i : \mu_S(x_i) \geq \alpha\}$ corresponding to all possible $\alpha \in (0, 1]$,
  - then we can uniquely reconstruct the corresponding membership function $\mu_S(x_i)$.
- The possibility for such a reconstruction follows from the fact that:
  - every real number r
  - is equal to the largest largest value α for which

$$r \geq \alpha.$$

- In particular, for every x_i, the value $\mu_S(x_i)$ is equal to the largest value α for which $\mu_S(x_i) \geq \alpha$.
- By definition of the α-cut, the inequality $\mu_S(x_i) \geq \alpha$ is equivalent to $x_i \in \mathbf{x}_i(\alpha)$.

68. From Alpha-Cuts to Membership Functions (cont-d)

- Thus, for every x_i, the value $\mu_S(x_i)$ can be reconstructed as the largest value α for which $x_i \in \mathbf{x}_i(\alpha)$.
- So, we can alternatively view a membership function as a nested family of α-cuts.

69. Possible Exam Questions

- What is an α-cut?
- How to select α? Give examples.
- What is the relation between α-cuts corresponding to different α?
- For $\mu(x) = \max(0, 1 - |x|)$ and $\alpha = 0.6$, what is the α-cut?
- Here, $1 - |x| \geq 0.6$ means $|x| \leq 1 - 0.6 = 0.4$, i.e., that $-0.4 \leq x \leq 0.4$.
- Thus, $\mathbf{x}(0.6) = [-0.4, 0.4]$.
- *For students to do on their own:* what is $\mathbf{x}(0.6)$ for $\mu(x) = \max(0, 1 - |x - 1|)$?

70. Fuzzy Numbers

- In most practical situations, the membership function:
  - starts with 0,
  - continuously increases until a certain value and then
  - continuously decreases to 0.
- Such membership function describe usual expert's expressions such as:
  - "small",
  - "medium",
  - "reasonably high",
  - "approximately equal to a with an error about σ", etc.

71. Fuzzy Numbers and Intervals

- Membership functions of this type are actively used in expert estimates of number-valued quantities.
- So, they are usually called *fuzzy numbers*.
- For a fuzzy number $\mu_i(x_i)$, every α-cut $\mathbf{x}_i(\alpha)$ is an interval.
- Thus, a fuzzy number can be viewed as:
  - a nested family of intervals $\mathbf{x}_i(\alpha)$
  - corresponding to different degrees of confidence α.

72. Possible Exam Questions

- What is a fuzzy number? Give examples.
- How to describe a fuzzy number in terms of α-cuts?

73. Simplest "And"- and "Or"-Operations: Reformulation in Terms of Sets and Alpha-Cuts

- The main formulas for fuzzy computations (i.e., for processing fuzzy data) were derived by using:
  - the "and"-operation $f_\&(d_1, d_2) = \min(d_1, d_2)$
  - and the "or"-operation $f_\vee(d_1, d_2) = \max(d_1, d_2)$.
- Let us reformulate these "and"- and "or"-operations in terms of α-cuts.
- Let us assume that we have two properties A and B which are described:
  - by the membership functions $\mu_A(x)$ and $\mu_B(x)$ and,
  - correspondingly, by the α-cuts

$$\mathbf{x}_A(\alpha) = \{x : \mu_A(x) \ge \alpha\} \text{ and } \mathbf{x}_B(\alpha) = \{x : \mu_B(x) \ge \alpha\}.$$

74. "And"- and "Or"-Operations in Terms of Alpha-Cuts

- If we use the simplest "and"-operation $f_\&(d_1, d_2) = \min(d_1, d_2)$, then:
  - the composite property $A \,\&\, B$ ("A and B")
  - is described by the membership function $\mu_{A\,\&\,B}(x) = \min(\mu_A(x), \mu_B(x))$.
- What are the α-cuts corresponding to this membership function $\mathbf{x}_{A\,\&\,B}(\alpha) = \{x : \mu_{A\,\&\,B}(x) \geq \alpha\}$?
- The minimum of two real numbers:
  - is greater than or equal to α if and only if
  - both of these numbers are greater than or equal to α.
- Thus, $\mu_{A\,\&\,B}(x) = \min(\mu_A(x), \mu_B(x)) \geq \alpha$ is equivalent to "$\mu_A(x) \geq \alpha$ and $\mu_B(x) \geq \alpha$".

75. "And"- and "Or"-Operations in Terms of Alpha-Cuts (cont-d)

- So, the set $\mathbf{x}_{A\,\&\,B}(\alpha)$ of all x for which $\mu_{A\,\&\,B}(x) = \min(\mu_A(x), \mu_B(x)) \ge \alpha$ is an intersection of:
  - the set of all x for which $\mu_A(x) \ge \alpha$ and
  - the set of all x for which $\mu_B(x) \ge \alpha$.
- In other words, for every α, we have

$$\mathbf{x}_{A\,\&\,B}(\alpha) = \mathbf{x}_A(\alpha) \cap \mathbf{x}_B(\alpha).$$

- Therefore:
  - to perform the simplest "and"-operation $f_\&(d_1, d_2) = \min(d_1, d_2)$,
  - we simply take the intersection of the corresponding α-cuts.

76. "And"- and "Or"-Operations in Terms of Alpha-Cuts (cont-d)

- This is a very natural operation.
- Indeed, for exactly defined properties, the set of all the elements which satisfy $A \,\&\, B$ is the intersection of:
  - the set of all elements which satisfy property A and
  - the set of all elements which satisfy property B.
- Similarly, for the simplest "or"-operation $f_\vee(d_1, d_2) = \max(d_1, d_2)$,
  - the composite property $A \vee B$ ("A or B")
  - is described by the membership function

$$\mu_{A \vee B}(x) = \max(\mu_A(x), \mu_B(x)).$$

- We want to find the α-cuts corresponding to this membership function

$$\mathbf{x}_{A \vee B}(\alpha) = \{x : \mu_{A \vee B}(x) \geq \alpha\}.$$

77. "And"- and "Or"-Operations in Terms of Alpha-Cuts (cont-d)

- The maximum of two real numbers:
  - is greater than or equal to α
  - if and only if one of these numbers is greater than or equal to α.
- Thus, $\mu_{A\vee B}(x) = \max(\mu_A(x), \mu_B(x)) \ge \alpha$ is equivalent to "$\mu_A(x) \ge \alpha$ or $\mu_B(x) \ge \alpha$".
- Hence:
  - the set $\mathbf{x}_{A\vee B}(\alpha)$ of x for which the condition $\mu_{A\vee B}(x) = \max(\mu_A(x), \mu_B(x)) \ge \alpha$ is satisfied
  - is the union of the set of all x for which $\mu_A(x) \ge \alpha$ and the set of all x for which $\mu_B(x) \ge \alpha$.
- In other words, for every α, we have

$$\mathbf{x}_{A\vee B}(\alpha) = \mathbf{x}_A(\alpha) \cup \mathbf{x}_B(\alpha).$$

78. "And"- and "Or"-Operations in Terms of Alpha-Cuts (cont-d)

- Therefore:
  - to perform the simplest "or"-operation $f_\vee(d_1, d_2) = \max(d_1, d_2)$,
  - we simply take the union of the corresponding α-cuts.
- This is also a very natural operation.
- Indeed, for exactly defined sets and properties: the set of all the elements which satisfy the property $A \vee B$:
  - is equal to the union of
  - the set of all elements which satisfy property A and
  - the set of all elements which satisfy property B.

79. Possible Exam Questions

- How to describe "and" in terms of α-cuts? Why is this description natural?
- How to describe "or" in terms of α-cuts? Why is this description natural?
- $\mu_1(x_1) = \max(0, 1 - |x_1|)$, $\mu_2(x_2) = \max(0, 1 - |x_2 - 1|)$, $y \stackrel{\text{def}}{=} x_1 \cup x_2$, $z \stackrel{\text{def}}{=} x_1 \cap x_2$, and $\alpha = 0.6$.
  - What is the α-cut for y?
  - What is the α-cut for or z?

80. Fuzzy Computations Can Be Reduced to Interval Computations: Derivation

- The main problem of fuzzy computation can be described as follows:
- We know an algorithm $y = f(x_1, \ldots, x_n)$ that relates:
  - the value of the desired difficult-to-estimate quantity y
  - with the values of easier-to-estimate auxiliary quantities $x_1, \ldots, x_n$.
- We also know, for every i from 1 to n, a membership function $\mu_i(x_i)$.
- This function describes the expert knowledge about the i-th input quantity x_i.
- Our objective is to compute the function

$$\mu(y) = \sup\{\min(\mu_1(x_1), \ldots, \mu_n(x_n)) : y = f(x_1, \ldots, x_n)\}.$$

81. Reduction to Intervals (cont-d)

- Let us now describe this relation in terms of α-cuts.
- This relation was first discovered and proved by Hung T. Nguyen in 1978.
- To describe this result in precise terms, let us first make some mathematics-related remarks.
- The function $y = f(x_1, \ldots, x_n)$ describes the relation between physical quantities.
- In physics, such a relation is usually continuous.
- Sometimes, we have a seemingly discontinuous transitions, e.g., in phase transitions when, e.g.:
  - the density of water changes into
  - a much smaller density of steam.

82. Reduction to Intervals (cont-d)

- However, it is not really a discontinuous transition.
- It is simply a very fast but still continuous one.
- In view of this observation, we will assume that the function $y = f(x_1, \ldots, x_n)$ is continuous.
- We will also assume the membership functions $\mu_i(x_i)$ are continuous.
- If we had exact knowledge, then continuity would make no sense.
- Indeed, then, the corresponding degree of confidence would abruptly go:
  - from 1 for possible values
  - to 0 for impossible ones,
  - without ever attaining any intermediate degrees.

83. Reduction to Intervals (cont-d)

- However, for fuzzy knowledge, continuity makes perfect sense:
  - if there is some degree of confidence that a value x_i is possible,
  - then it makes sense to assume that values close to x_i are possible too,
  - with a similar degree of belief.
- In practice, as we mentioned earlier, membership functions are indeed usually continuous.
- It is important to mention that:
  - for continuous membership functions $\mu_i(x_i)$,
  - the α-cuts $\{x_i : \mu_i(x_i) \geq \alpha\}$ are closed sets (i.e., sets which contain all their limit points).

84. Reduction to Intervals (cont-d)

- Finally, we require that for every i and for every $\alpha > 0$, the α-cut is a compact set.
- For real numbers:
  - since we have already assumed that the α-cuts $\{x_i : \mu_i(x_i) \geq \alpha\}$ are closed sets,
  - it is sufficient to require that these sets are bounded.
- This is true, e.g., if we assume that all the membership functions correspond to fuzzy numbers.
- In this case, all α-cuts are intervals.
- Suppose that we know the α-cuts $\mathbf{x}_i(\alpha)$ corresponding to the inputs.
- We want to find the α-cuts $\mathbf{y}(\alpha)$ corresponding to the output.

85. Reduction to Intervals (cont-d)

- By definition of an α-cut, $y \in \mathbf{y}(\alpha)$ means that $\mu(y) \ge \alpha$, i.e., that

$$\sup\{\min(\mu_1(x_1), \ldots, \mu_n(x_n)) : y = f(x_1, \ldots, x_n)\} \ge \alpha.$$

- By definition of the supremum, this means that:
  - for every integer $k > 2/\alpha$,
  - there exists a tuple $(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$ for which $y = f(x_1^{(k)}, \ldots, x_n^{(k)})$ and

$$\min(\mu_1(x_1^{(k)}), \mu_2(x_2^{(k)}), \ldots) \ge \alpha - 1/k.$$

- The minimum of several numbers is $\ge \alpha - 1/k$ if and only if all these numbers are $\ge \alpha - 1/k$, i.e.,

$$\mu_i(x_i^{(k)}) \ge \alpha - 1/k \text{ for all } i.$$

- Since $k > 2/\alpha$, we have $1/k < \alpha/2$ and $\alpha - 1/k > \alpha/2$.

86. Reduction to Intervals (cont-d)

- Thus, for each i and all k, the value $x_i^{(k)}$ belongs to the compact $(\alpha/2)$-cut $\mathbf{x}_i(\alpha/2)$.
- The tuples $(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$ belong to the compact set
$$\mathbf{x}_1(\alpha/2) \times \mathbf{x}_2(\alpha/2) \times \ldots \times \mathbf{x}_n(\alpha/2).$$
- Thus, the sequence of these tuples has a convergent subsequence converging to some tuple $(x_1, x_2, \ldots, x_n)$.
- Since both f and μ_i are continuous, for this limit tuple, we get $y = f(x_1, \ldots, x_n)$ and $\mu_i(x_i) \geq \alpha$.
- In other words, every element $y \in \mathbf{y}(\alpha)$ can be represented as $y = f(x_1, \ldots, x_n)$ for some values $x_i \in \mathbf{x}_i(\alpha)$.

87. Reduction to Intervals (cont-d)

- Conversely, if $x_i \in \mathbf{x}_i(\alpha)$ and $y = f(x_1, \ldots, x_n)$, then $\mu_i(x_i) \geq \alpha$.
- So, $\min(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)) \geq \alpha$, hence

$$\sup\{\min(\mu_1(x_1), \ldots, \mu_n(x_n)) : y = f(x_1, \ldots, x_n)\} \geq \alpha.$$

- Thus, $\mu(y) \geq \alpha$ and $y \in \mathbf{y}(\alpha)$.
- So, the desired α-cut $\mathbf{y}(\alpha)$ consists of exactly values $y = f(x_1, \ldots, x_n)$ for $x_i \in \mathbf{x}_i(\alpha)$:

$$\mathbf{y}(\alpha) = \{f(x_1, \ldots, x_n) : x_1 \in \mathbf{x}_1(\alpha), \ldots, x_n \in \mathbf{x}_n(\alpha)\}.$$

- For fuzzy numbers, all α-cuts are intervals, so this is exactly the interval computations range:

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \ldots, \mathbf{x}_n(\alpha)).$$

88. Fuzzy Computations Can Be Reduced to Interval Computations: Conclusion

- If the inputs $\mu_i(x_i)$ are fuzzy numbers and the function $y = f(x_1, \ldots, x_n)$ is continuous, then
  - for each α, the α-cut $\mathbf{y}(\alpha)$ of y is equal to
  - the range of possible values of $f(x_1, \ldots, x_n)$ as x_i ranges over $\mathbf{x}_i(\alpha)$ for all i:

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \ldots, \mathbf{x}_n(\alpha)).$$

- Thus, from the computational point of view,
  - the problem of processing data under fuzzy uncertainty
  - can be reduced to several problems of data processing under interval uncertainty,
  - as many problems as there are α-levels.

89. Fuzzy Computations Can Be Reduced to Interval Computations: Conclusion (cont-d)

- So, fuzzy computations can be reduced to interval computations.
- As we have mentioned, this is not just a theoretical observation.
- This is exactly how fuzzy data processing is usually performed.
- This is why interval computations techniques are explained in fuzzy textbooks.

90. Possible Exam Questions

- Assume that $u = x_1 + x_2$, $\mu_1(x_1) = \max(0, 1 - |x_1|)$, and $\mu_2(x_2) = \max(0, 1 - |x_2 - 1|)$. What is $\mathbf{y}(0.6)$?
- We have $\mathbf{y}(0.6) = \mathbf{x}_1(0.6) + \mathbf{x}_2(0.6)$.
- We already know that $\mathbf{x}_1(0.6) = [-0.4, 0.4]$ and $\mathbf{x}_2(0.6) = [0.6, 1.4]$.
- So, $\mathbf{y}(0.6) = [-0.4, 0.4] + [0.6, 1.4] = [0.2, 1.8]$.
- For students to do on their own: what is $y = x_1 - x_2$? what if $\alpha = 0.7$?

91. Intervals are Necessary to Describe Degrees of Belief

- Above, we described an idealized situation, in which we can describe degrees of belief by exact real numbers.
- In practice, the situation is more complicated.
- Reason: experts cannot describe their degrees of belief precisely.
- Indeed, let us start by reviewing the above-described methods of eliciting degrees of belief.
  - if an expert describes his or her degree of belief by selecting, e.g., 8 on a scale from 0 to 10,
  - this does not mean that his or her degree of belief is exactly 0.8
  - if instead, we ask him or her to select on a scale from 0 to 9,

- then whatever he or she chooses, after dividing it by 9, we will never get 0.8.

- If an expert chooses a value 8 on a 0 to 10 scale, then:
  - the only thing that we know about the expert's degree of belief
  - is that it is closer to 8 than to 7 or to 9,
  - i.e., that this degree of belief belongs to the *interval* $[0.75, 0.85]$.

- Another possible source of interval uncertainty is when we have *several* experts, and their estimates differ:
  - if, e.g., two equally good experts point to 7 and 8,
  - then, if we are cautious, we would rather describe the resulting degree of belief as the interval $[0.7, 0.8]$
  - or, in view of the above remark, as the interval $[0, 65, 0.85]$.

- If we determine the degree of belief by polling, then the same argument shows that the resulting numbers are not precise:
  - if 8 out of 10 experts voted for A,
  - then we cannot say that the actual degree of belief is exactly 0.8,
  - because, if we repeated this procedure with 9 experts,
  - we will never get exactly 0.8.
- In this case, there are two other sources of uncertainty:
  - First, picking experts is sort of a random procedure,
  - so, the result of voting is a statistical estimate that is not precise,
  - just like a statistical frequency estimate of probability;
  - better description will be to give an *interval* of possible values of $d(A)$.

- The polling method of estimating the degree of belief is based on the assumption that an expert can always tell whether he believes in a given statement S or not.
- Then, we take the ratio $d(S) = N(S)/N$ of the number $N(S)$ of experts who believe in S to the total number N of experts as the desired estimate.
- For $\neg S$, we thus have $N(\neg S) = N - N(S)$, so $d(\neg S) = N(\neg S)/N = 1 - d(S)$.
- In reality, an expert is often unsure about S; in this case:
  - instead of dividing the experts into two categories: those who believe in S and those who do not,
  - we must divide them into *three* categories:
    * those who believe that S is true (we will denote their number by $N(S)$),
    * those who believe that S is false (we will denote their number by $N(\neg S)$), and

    * those who do not have the definite opinion about S; there are $N - N(S) - N(\neg S)$ of them.

- In this situation, one number is not sufficient to describe the experts' degree of belief in S.
- We need at least two numbers.
- There are two ways to describe it.
- We can describe the degree of belief in S as $d(S) = N(S)/N$ and the degree of belief in $\neg S$ as $d(\neg S) = N(\neg S)/N$.
- These two numbers must satisfy the condition $d(S) + d(\neg S) \le 1$.
- This description is known as *intuitionistic fuzzy logic*.
- Alternatively, we can describe:
  - the degree of belief $d(S)$ in S and

- the degree of *plausibility* of S estimating as the fraction of experts who do not consider S impossible, i.e., as $pl(S) = 1 - d(\neg S)$.

- Thus, we get an interval $[d(S), pl(S)]$.

- This representation corresponds to the so-called *Dempster-Shafer formalism*.

- So, to describe degrees of belief adequately, we must use *intervals* instead of real numbers.

92. Possible Exam Questions

- Why do we need interval-valued degrees of belief?
- If an expert marked his/her degree of confidence as 4 on a 0 to 5 scale, what is the corresponding interval?
- What is intuitionistic fuzzy logic?
- What is Dempster-Shafer approach?
- If out of 10 experts, 3 say "yes", 4 say "no", and 3 are undecided, how will this be represented:
  - in intuionistic fuzzy logic?
  - in Dempster-Shafer approach?

93. Interval Computations for Processing Interval-Valued Degrees of Belief

- For an expert system with interval-valued degrees of belief, the following problem arises.
- Suppose that we have an expert system whose knowledge base consists of statements $S_1, \ldots, S_N$.
- We have an algorithm $f(Q, d_1, \ldots, d_N)$ (called *inference engine*) that, for any given query Q:
  - transforms the degrees of belief $d(S_1), \ldots, d(S_N)$ in the statements from the knowledge base
  - into a degree of belief in Q:

$$d(Q) = f(Q, d(S_1), \ldots, d(S_N)).$$

- For example, if $Q = S_1 \& S_2$, then $N = 2$ and $f(d_1, d_2) = f_\&(d_1, d_2)$.

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Need to Process Fuzzy . . .
Degrees of Belief
Membership Functions
Need for "And"- and . . .
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94. Interval Computations for Processing Interval-Valued Degrees of Belief (cont-d)

- Suppose now that we know only the *intervals* $\mathbf{d}(S_1)$, $\ldots$, $\mathbf{d}(S_N)$ that contain the desired degree of belief.
- Then, the degree of belief in Q can take any value from:

$$f(Q, \mathbf{d}(S_1), \ldots, \mathbf{d}(S_N)) \stackrel{\text{def}}{=} \{f(Q, d_1, \ldots, d_N) \mid d_i \in \mathbf{d}(S_i)\}.$$

- Computing such an interval is a typical problem of *interval computations*.
- Since $f_\&$ and $f_\vee$ are increasing in both arguments:

$$f_\&([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]) = [f_\&(\underline{x}, \underline{y}), f_\&(\overline{x}, \overline{y})],$$

$$f_\vee([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]) = [f_\vee(\underline{x}, \underline{y}), f_\vee(\overline{x}, \overline{y})].$$

- For example,

$$\min([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]) = [\min(\underline{x}, \underline{y}), \min(\overline{x}, \overline{y})],$$

$$\max([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]) = [\max(\underline{x}, \underline{y}), \max(\overline{x}, \overline{y})].$$

95. Possible Exam Questions

- For two statements S_1 and S_2, we have $\mathbf{d}_1 = [0.5, 0.7]$ and $\mathbf{d}_2 = [0.3, 0.4]$.
  - What is the degree of confidence in $S_1 \,\&\, S_2$?
  - What is the degree of confidence in in $S_1 \vee S_2$?

96. Conclusion

- We have explained intrinsic and useful relations between interval computing and fuzzy data processing.
- The main relation is that a fuzzy set (membership function) can be viewed as a nested family of intervals.
- Its α-cuts corresponding to different levels of uncertainty α.
- From the computational viewpoint:
  - fuzzy data processing can be (and usually is)
  - reduced to level-by-level interval computations with the corresponding α-cuts.

97. Conclusion (cont-s)

- Another relation comes from the fact that
  - it is usually difficult to describe experts' degrees of certainty by exact real numbers;
  - a more adequate description of expert's uncertainty is by an interval.
- Processing interval-valued degrees of uncertainty also requires interval computations.