Bisection

Problem. We know the values a and b for which $f(a) < 0$ and $f(b) > 0$. We need to find the *root*, i.e., the value x for which $f(x) = 0$.

We want to find x with a given accuracy $\varepsilon > 0$, i.e., we want to find the value x_0 which is ε-close to the root.

Solution. This is an iterative algorithm. At each point of this algorithm, we have two value $\underline{x} < \overline{x}$ for which $f(\underline{x}) < 0$ and $f(\overline{x}) > 0$.

- We start with $\underline{x} = a$ and $\overline{x} = b$.
- We stop when $\overline{x} - \underline{x} \le 2\varepsilon$, then we return $x_0 = \dfrac{\underline{x} + \overline{x}}{2}$.

If $\overline{x} - \underline{x} > 2\varepsilon$, then we take $m = \dfrac{\underline{x} + \overline{x}}{2}$ and compute $f(m)$.

- If $f(m) < 0$, then we replace $\underline{x}$ with m: $\underline{x} := m$.
- If $f(m) > 0$, then we replace $\overline{x}$ with m: $\overline{x} := m$.
- In the rare case when $f(m) = 0$, we find the root, so we stop.

Example. Let us use this method to find $\sqrt{2}$, i.e., to find the value x for which $f(x) = x^2 - 2$ is equal to 0. Let us find it with accuracy $\varepsilon = 0.1$.

- We start with $a = 0$ and $b = 2$, since

  $$f(0) = 0^2 - 2 = -2 < 0$$

  and

  $$f(2) = 2^2 - 2 = 4 - 2 = 2 > 0.$$

  So, the initial interval $[\underline{x}, \overline{x}]$ is $[0, 2]$.

- The width $2 - 0 = 2$ of the interval $[0, 2]$ is larger than $2\varepsilon = 0.2$, so we find the value $f(m)$ for the midpoint $m = 1$. For this midpoint,

  $$f(1) = 1^2 - 2 = 1 - 2 = -1 < 0,$$

  so we replace $\underline{x}$ with 1, and conclude that x is in the interval $[1, 2]$.

- The width $2 - 1 = 1$ of the interval $[1, 2]$ is larger than $2\varepsilon = 0.2$, so we find the value $f(m)$ for the midpoint $m = 1.5$. For this midpoint,

  $$f(1.5) = 1.5^2 - 2 = 2.25 - 2 = 0.25 > 0,$$

  so we replace $\overline{x}$ with 1.5, and conclude that x is in the interval $[1, 1.5]$.

- The width $1.5 - 1 = 0.5$ of the interval $[1, 1.5]$ is larger than $2\varepsilon = 0.2$, so we find the value $f(m)$ for the midpoint $m = 1.25$. For this midpoint,

  $$f(1.25) = 1.25^2 - 2 = 1.5625 - 2 = -0.4375 < 0,$$

  so we replace $\underline{x}$ with 1.25, and conclude that x is in the interval $[1.25, 1.5]$.

- The width $1.5-1.25 = 0.25$ of the interval $[1.25, 1.5]$ is larger than $2\varepsilon = 0.2$, so we find the value $f(m)$ for the midpoint $m = 1.375$. For this midpoint,

  $$f(1.375) = 1.375^2 - 2 = 1.88\ldots - 2 < 0,$$

  so we replace $\underline{x}$ with 1.375, and conclude that x is in the interval $[1.375, 1.5]$.

The width $1.375 - 1.25 = 0.125$ of the interval $[1.375, 1.5]$ is smaller than $2\varepsilon = 0.2$, so we return the midpoint of this interval as the desired answer:

$$x_0 = \frac{1.375 + 1.5}{2} = 1.4375.$$

Comment. This number is indeed 0.1-close to $\sqrt{2} = 1.41\ldots$