How Calculus Helps Find the Range: Functions of One Variable

Problem: reminder.

- We are given a function $y = f(x)$ and a range $[\underline{x}, \overline{x}]$.
- We want to find the range
$$[\underline{y}, \overline{y}] = \{f(x) : x \in [\underline{x}, \overline{x}]\}$$
of all possible values $y = f(x)$ when $x \in [\underline{x}, \overline{x}]$.

Idea. In this case,
$$\underline{y} = \min_{x \in [\underline{x}, \overline{x}]} f(x); \quad \overline{y} = \max_{x \in [\underline{x}, \overline{x}]} f(x).$$

Reminder. According to calculus, maximum and minimum of a function $f(x)$ on an interval $[\underline{x}, \overline{x}]$ are attained:

- either at one of the endpoints,
- or at a point inside the interval; in this case, the derivative $f'(x) = 0$ is equal to 0.

Resulting algorithm.

- Find all the values $x \in [\underline{x}, \overline{x}]$ at which $f'(x) = 0$.
- Compute the values of the function $f(x)$ at the endpoints $\underline{x}$ and $\overline{x}$ of the interval and at all the points $x \in [\underline{x}, \overline{x}]$ (if any) at which $f'(x) = 0$.
- The smallest of these values of the function is $\underline{y}$, the largest is $\overline{y}$.

How to differentiate: reminder. If x is the unknown, then:

- for a constant c, we have $c' = 0$;
- for a linear function $a \cdot x$, we have $(a \cdot x)' = a$;
- for the sum, $(a + b)' = a' + b'$;

- for the product, $(a \cdot b)' = a' \cdot b + a \cdot b'$;
- for the ratio,
$$\left(\frac{a}{b}\right)' = \frac{a' \cdot b - a \cdot b'}{b^2};$$
- for the power function, $(x^n)' = n \cdot x^{n-1}$;
- for the composition, $(f(g(x))' = f'(g(x)) \cdot g'(x)$;
- for special functions:
$$(\exp(x))' = \exp(x),\ (\ln(x))' = 1/x,$$
$$(\sin(x))' = \cos(x),\ (\cos(x))' = -\sin(x).$$

Example. Let us use calculus to find the range of a function
$$f(x) = (2 + x) \cdot (6 - x)$$
on the interval $[0, 6]$.

The derivative of $2 + x$ is 1, the derivative of $6 - x$ is -1, so by using the formula for the derivative of the product, we conclude that
$$f'(x) = 1 \cdot (6 - x) + (2 + x) \cdot (-1) = 6 - x - 2 - x = 4 - 2x.$$
Thus, this derivative is equal to 0 when $4 - 2x = 0$, i.e., when $x = 2$.

So, to find the range of the function, it is sufficient to compute its values at the endpoints $x = 0$ and $x = 6$ and at the point $x = 2$ where the derivative is equal to 0.

- For $x = 0$, we have $f(0) = 2 \cdot 6 = 12$.
- For $x = 6$, we have $f(6) = 8 \cdot 0 = 0$.
- For $x = 2$, we have $f(2) = 4 \cdot 4 = 16$.

The smallest of these three values is 0, the largest is 16, so the range is $[0, 16]$.