

Decision Making under Interval Uncertainty

How decisions are made. Usually, we gauge the quality of each alternative by a number x_i – the larger the better – and we select the best alternative, i.e., an alternative i :

- for which $x_i \geq x_j$ for all $j \neq i$, or, equivalently,
- for which $x_i = \max_j x_j$.

Case of interval uncertainty: formulation of the problem. In many practical situations:

- we do not know the exact value of the quantity x_i ;
- we only know the interval $[\underline{x}_i, \bar{x}_i]$.

Alternatives which are guaranteed to be optimal: analysis of the problem. Sometimes, there is an alternative i that is *guaranteed* to be optimal, i.e., for which $x_i \geq x_j$ for all other alternatives $j \neq i$ and for all possible values $x_i \in [\underline{x}_i, \bar{x}_i]$ and $x_j \in [\underline{x}_j, \bar{x}_j]$.

The inequality $x_i \geq x_j$ holds for all possible values x_i and x_j if the smallest possible value \underline{x}_i is larger than or equal to the largest possible value \bar{x}_j . So, an alternative i is guaranteed to be optimal if $\underline{x}_i \geq \bar{x}_j$ for all $j \neq i$.

This, in turn, is equivalent to \underline{x}_i being larger than or equal to the largest of the values \bar{x}_j . So, we arrive at the following algorithm.

Alternatives which are guaranteed to be optimal: algorithm. An alternative i is guaranteed to be optimal if

$$\underline{x}_i \geq \max_{j \neq i} \bar{x}_j.$$

Example. Let $[\underline{x}_1, \bar{x}_1] = [0, 5]$, $[\underline{x}_2, \bar{x}_2] = [6, 10]$, and $[\underline{x}_3, \bar{x}_3] = [1, 4]$. In this case,

$$\underline{x}_2 = 6 \geq \max_{j \neq 2} \bar{x}_j = \max(\bar{x}_1, \bar{x}_3) = \max(5, 4) = 5.$$

So, the alternative 2 is guaranteed to be optimal.

Alternatives which are possibly optimal: analysis of the problem. An alternative i is possibly optimal if there are some values $x_j \in [\underline{x}_j, \bar{x}_j]$ for which $x_i \geq x_j$ for all $j \neq i$. This implies that

$$\bar{x}_i \geq x_i \geq x_j \geq \underline{x}_j,$$

hence $\bar{x}_i \geq \underline{x}_j$ for all $j \neq i$.

Vice versa, if $\bar{x}_i \geq \underline{x}_j$ for all $i \neq j$, then we take $x_i = \bar{x}_i$ and $x_j = \underline{x}_j$ and we will have $x_i \geq x_j$ for all $j \neq i$.

The condition that $\bar{x}_i \geq \underline{x}_j$ for all j is equivalent to \bar{x}_i being greater than or equal to the largest of the values \underline{x}_j . Thus, we arrive at the following algorithm:

Alternatives which are possible optimal: algorithm. An alternative i is possibly optimal if

$$\bar{x}_i \geq \max_{j \neq i} \underline{x}_j.$$

Example. Let $[\underline{x}_1, \bar{x}_1] = [0, 5]$, $[\underline{x}_2, \bar{x}_2] = [4, 5]$, and $[\underline{x}_3, \bar{x}_3] = [1, 3]$. In this case,

$$\max_j \underline{x}_j = \max(0, 4, 1) = 4,$$

so the above inequality is satisfied for $i = 1$ and $i = 2$. Thus, alternatives 1 and 2 are possibly optimal.

Hurwicz approach: description. If we need to select a single alternative, then the usual approach is to use the so-called Hurwicz optimism-pessimism criterion, Namely, we select the value $\alpha \in [0, 1]$ and then select an alternative i for which the value $x_i = \alpha \cdot \bar{x}_i + (1 - \alpha) \cdot \underline{x}_i$ is the largest.

The value $\alpha = 1$ means that compare the most optimistic values \bar{x}_i , the value $\alpha = 0$ means that we compare the most pessimistic values \underline{x}_i . Intermediate values α means that we take both the optimistic and the pessimistic values into account.

Hurwicz approach: idea. Let us illustrate this approach on the example of three intervals $[\underline{x}_1, \bar{x}_1] = [0, 5]$, $[\underline{x}_2, \bar{x}_2] = [4, 5]$, and $[\underline{x}_3, \bar{x}_3] = [1, 3]$ and values $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1$. For each α , the alternative with the largest value of x_i is underlined:

	$i = 1$	$i = 2$	$i = 3$
$\alpha = 0$	0	<u>4</u>	1
$\alpha = 0.5$	2.5	<u>4.5</u>	2
$\alpha = 1$	<u>5</u>	<u>5</u>	3