Example of Interval Computations

1 Problem

Problem. We want to estimate the range of the function

$$f(x_1, x_2) = x_1^2 + x_1 \cdot x_2 + 2x_2^2 + 1$$

when $x_1 \in [\underline{x}_1, \overline{x}_1] = [-0.3, 0.1]$ and $x_2 \in [\underline{x}_2, \overline{x}_2] = [-0.1, 0.3]$.

Original problem: preliminary computations. In this case,

$$\widetilde{x}_1 = \frac{\underline{x}_1 + \overline{x}_1}{2} = \frac{(-0.3) + 0.1}{2} = \frac{-0.2}{2} = -0.1$$

and

$$\widetilde{x}_2 = \frac{\underline{x}_2 + \overline{x}_2}{2} = \frac{(-0.1) + 0.3}{2} = \frac{0.2}{2} = 0.1.$$

Substituting these values into the expression for the function $f(x_1, x_2)$, we get the result of data processing:

$$\widetilde{y} = f(\widetilde{x}_1, \widetilde{x}_2) = f(-0.1, 0.1) = (-0.1)^2 + (-0.1) \cdot 0.1 + 2 \cdot 0.1^2 + 1 =$$

$$0.01 - 0.01 + 0.02 + 1 = 1.02.$$

General algorithm: reminder.

- First, we check for monotonicity with respect to each variable. If there is monotonicity with respect to some of the variables, we reduce the problem to two problem with fewer variables.
- If after this, we still do not have the exact range, we use the centered form.
- If we want to get a more accurate estimate, we bisect one the intervals, namely, one for which the product $|c_i| \cdot \Delta_i$ is the largest, where, as before, $c_i \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_i}(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, and $\Delta_i = \dfrac{\overline{x}_i - \underline{x}_i}{2}$, and repeat the same process again for the two resulting problems.

2 First Stage: Before Bisection

Checking monotonicity. Let us first check whether the function is monotonic with respect to x_1. Here,

$$\frac{\partial f}{\partial x_1} = 2x_1 + x_2.$$

We use straightforward interval computations to find the range of this derivative:

$$\frac{\partial f}{\partial x_1}([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) = \frac{\partial f}{\partial x_1}([-0.3, 0.1], [-0.1, 0.3]) =$$

$$2 \cdot [-0.3, 0.1] + [-0.1, 0.3] =$$

$$[-0.6, 0.2] + [-0.1, 0.3] = [-0.7, 0.5].$$

This interval contains both positive and negative values, so the function is not monotonic with respect to x_1 on the box $[-0.3, 0.1] \times [-0.1, 0.3]$.

So, we need to check monotonicity with respect to x_2. Here:

$$\frac{\partial f}{\partial x_2} = x_1 + 4x_2.$$

We use straightforward interval computations to find the range of this derivative:

$$\frac{\partial f}{\partial x_2}([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) = \frac{\partial f}{\partial x_2}([-0.3, 0.1], [-0.1, 0.3]) =$$

$$[-0.3, 0.1] + 4 \cdot [-0.1, 0.3] =$$

$$[-0.3, 0.1] + [-0.4, 1.2] = [-0.7, 1.3].$$

This interval contains both positive and negative values, so the function is not monotonic with respect to x_2 on the box $[-0.3, 0.1] \times [-0.1, 0.3]$.

Using centered form. Since the function is not monotonic, we have to use the centered form, and compute the following enclosure for the range:

$$[\underline{Y}, \overline{Y}] = \widetilde{y} + \frac{\partial f}{\partial x_1}([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) \cdot [-\Delta_1, \Delta_1] + \frac{\partial f}{\partial x_2}([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) \cdot [-\Delta_2, \Delta_2].$$

In our case:

$$\Delta_1 = \frac{\overline{x}_1 - \underline{x}_1}{2} = \frac{0.1 - (-0.3)}{2} = \frac{0.4}{2} = 0.2$$

and

$$\Delta_2 = \frac{\overline{x}_2 - \underline{x}_2}{2} = \frac{0.3 - (-0.1)}{2} = \frac{0.4}{2} = 0.2.$$

Thus,

$$[\underline{Y}, \overline{Y}] = 1.02 + [-0.7, 0.5] \cdot [-0.2, 0.2] + [-0.7, 1.3] \cdot [-0.2, 0.2] =$$

$$1.02 + [-0.14, 0.14] + [-0.26, 0.26] =$$

$$[0.88, 1.16] + [-0.26, 0.26] = [0.62, 1.42].$$

Resulting enclosure: $[\underline{Y}, \overline{Y}] = [0.62, 1.42]$.

3 Let Us Use Bisection

Which side to bisect? Here,

$$c_1 = \frac{\partial f}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) = 2\widetilde{x}_1 + \widetilde{x}_2 = 2 \cdot (-0.1) + 0.1 = -0.2 + 0.1 = -0.1$$

and

$$c_2 = \frac{\partial f}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_1 + 4\widetilde{x}_2 = (-0.1) + 4 \cdot 0.1 = -0.1 + 0.4 = 0.3.$$

Thus,

$$|c_1| \cdot \Delta_1 = |-0.1| \cdot 0.2 = 0.1 \cdot 0.2 = 0.02$$

and

$$|c_2| \cdot \Delta_2 = |0.3| \cdot 0.2 = 0.06.$$

Here, $|c_2| \cdot \Delta_2 > |c_1| \cdot \Delta_1$, so we bisect by x_2.

How we bisect: reminder. In general, bisection means that we replace the original interval $[\underline{x}_i, \overline{x}_i]$ by its midpoint $\widetilde{x}_i$ into two equal subintervals $[\underline{x}_i, \widetilde{x}_i]$ and $[\widetilde{x}_i, \overline{x}_i]$.

Original problem: how we bisect. In our case, $i = 2$, so we replace the original interval $[\underline{x}_2, \overline{x}_2] = [-0.1, 0.3]$ with two subintervals $[-0.1, 0.1]$ and $[0.1, 0.3]$. Thus, we replace the original problem with the following two subproblems:

Subproblem 1. Estimate the range of the function

$$f(x_1, x_2) = x_1^2 + x_1 \cdot x_2 + 2x_2^2 + 1$$

when $x_1 \in [\underline{x}_1, \overline{x}_1] = [-0.3, 0.1]$ and $x_2 \in [\underline{x}_2, \overline{x}_2] = [-0.1, 0.1]$.

Subproblem 2. Estimate the range of the function

$$f(x_1, x_2) = x_1^2 + x_1 \cdot x_2 + 2x_2^2 + 1$$

when $x_1 \in [\underline{x}_1, \overline{x}_1] = [-0.3, 0.1]$ and $x_2 \in [\underline{x}_2, \overline{x}_2] = [0.1, 0.3]$.

Once we find the enclosures for these two problems, the union of these two enclosures will be an enclosure for our original problem. Let us start with Subproblem 1.

Subproblem 1: preliminary computations. In this case, we did not change the interval for x_1, so we still have $\widetilde{x}_1 = -0.1$. Here,

$$\widetilde{x}_2 = \frac{\underline{x}_2 + \overline{x}_2}{2} = \frac{(-0.1) + 0.1}{2} = \frac{0}{2} = 0.$$

Substituting these values into the expression for the function $f(x_1, x_2)$, we get the result of data processing:

$$\widetilde{y} = f(\widetilde{x}_1, \widetilde{x}_2) = f(-0.1, 0) = (-0.1)^2 + (-0.1) \cdot 0 + 2 \cdot 0^2 + 1 = 0.01 + 1 = 1.01.$$

Subproblem 1: checking monotonicity. Let us first check whether the function is monotonic with respect to x_1. We use straightforward interval computations to find the range of this derivative:

$$\frac{\partial f}{\partial x_1}([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) = \frac{\partial f}{\partial x_1}([-0.3, 0.1], [-0.1, 0.1]) =$$

$$2 \cdot [-0.3, 0.1] + [-0.1, 0.1] =$$

$$[-0.6, 0.2] + [-0.1, 0.1] = [-0.7, 0.3].$$

This interval contains both positive and negative values, so the function is not monotonic on the box $[-0.3, 0.1] \times [-0.1, 0.3]$.

So, we need to check monotonicity with respect to x_2. We use straightforward interval computations to find the range of this derivative:

$$\frac{\partial f}{\partial x_2}([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) = \frac{\partial f}{\partial x_2}([-0.3, 0.1], [-0.1, 0.1]) =$$

$$[-0.3, 0.1] + 4 \cdot [-0.1, 0.1] =$$

$$[-0.3, 0.1] + [-0.4, 0.4] = [-0.7, 0.5].$$

This interval contains both positive and negative values, so the function is not monotonic on the box $[-0.3, 0.1] \times [-0.1, 0.3]$.

Subproblem 1: using centered form. Since the function is not monotonic, we have to use the centered form. In our case, since we did not change the interval for x_1, we still have $\Delta_1 = 0.2$, and

$$\Delta_2 = \frac{\overline{x}_2 - \underline{x}_2}{2} = \frac{0.1 - (-0.1)}{2} = \frac{0.2}{2} = 0.1.$$

Thus,

$$[\underline{Y}, \overline{Y}] = 1.01 + [-0.7, 0.3] \cdot [-0.2, 0.2] + [-0.7, 0.5] \cdot [-0.1, 0.1] =$$

$$1.01 + [-0.14, 0.14] + [-0.07, 0.07] = [0.87, 1.15] + [-0.07, 0.07] = [0.80, 1.22].$$

Subproblem 2: preliminary computations. In this case, we did not change the interval for x_1, so we still have $\widetilde{x}_1 = -0.1$. Here,

$$\widetilde{x}_2 = \frac{\underline{x}_2 + \overline{x}_2}{2} = \frac{(-0.1) + 0.1}{2} = \frac{0}{2} = 0.$$

Substituting these values into the expression for the function $f(x_1, x_2)$, we get the result of data processing:

$$\widetilde{y} = f(\widetilde{x}_1, \widetilde{x}_2) = f(-0.1, 0) = (-0.1)^2 + (-0.1) \cdot 0 + 2 \cdot 0^2 + 1 = 0.01 + 1 = 1.01.$$

Subproblem 2: checking monotonicity. Let us first check whether the function is monotonic with respect to x_1. We use straightforward interval computations to find the range of this derivative:

$$\frac{\partial f}{\partial x_1}([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) = \frac{\partial f}{\partial x_1}([-0.3, 0.1], [-0.1, 0.1]) =$$

$$2 \cdot [-0.3, 0.1] + [0.1, 0.3] =$$

$$[-0.6, 0.2] + [0.1, 0.3] = [-0.5, 0.5].$$

This interval contains both positive and negative values, so the function is not monotonic on the box $[-0.3, 0.1] \times [-0.1, 0.3]$.

So, we need to check monotonicity with respect to x_2. We use straightforward interval computations to find the range of this derivative:

$$\frac{\partial f}{\partial x_2}([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]) = \frac{\partial f}{\partial x_2}([-0.3, 0.1], [-0.1, 0.1]) =$$

$$[-0.3, 0.1] + 4 \cdot [0.1, 0.3] =$$

$$[-0.3, 0.1] + [0.4, 1.2] = [0.1, 1.3].$$

All the values from this interval are positive, so the function is increasing with respect to x_2.

Subproblem 2: let us use monotonicity. Since the function is increasing with respect to x_2:

- To find the upper bound $\overline{Y}$ on this box, it is sufficient to consider the largest possible value of the variable x_2, i.e., the value $x_2 = \overline{x}_2 = 0.3$. For this value x_2, the function has the form

$$f^+(x_1) = x_1^2 + 0.3 \cdot x_1 + 0.3^2 + 1 = x_1^2 + 0.3 \cdot x_1 + 1.09.$$

- To find the lower bound $\overline{Y}$ on this box, it is sufficient to consider the smallest possible value of the variable x_2, i.e., the value $x_2 = \underline{x}_2 = 0.1$. For this value x_2, the function has the form

$$f^-(x_1) = x_1^2 + 0.1 \cdot x_1 + 0.1^2 + 1 = x_1^2 + 0.1 \cdot x_1 + 1.01.$$

In other words, we reduced the problem of estimating the range of the original function $f(x_1, x_2)$ to following two subproblems with fewer variables.

Subproblem 2.1. Estimate the range of the function

$$f^+(x_1) = x_1^2 + 0.3 \cdot x_1 + 1.09$$

when $x_1 \in [\underline{x}_1, \overline{x}_1] = [-0.3, 0.1]$.

Subproblem 2.2. Estimate the range of the function

$$f^-(x_1) = x_1^2 + 0.1 \cdot x_1 + 1.01$$

when $x_1 \in [\underline{x}_1, \overline{x}_1] = [-0.3, 0.1]$.

We will use the same technique to solve these two subproblems.

Subproblem 2.1: preliminary computations. In this case, still $\widetilde{x}_1 = -0.1$. Substituting these values into the expression for the function $f^+(x_1)$, we get the result of data processing:

$$\widetilde{y} = f^+(\widetilde{x}_1) = f^+(-0.1) = (-0.1)^2 + 0.3 \cdot (-0.1) + 1.09 = 0.01 - 0.03 + 1.09 = 1.07.$$

Subproblem 2.1: checking monotonicity. Here,

$$\frac{\partial f^+}{\partial x_1} = 2x_1 + 0.3.$$

We use straightforward interval computations to find the range of this derivative:

$$\frac{\partial f}{\partial x_1}([\underline{x}_1, \overline{x}_1]) = \frac{\partial f}{\partial x_1}([-0.3, 0.1]) =$$

$$2 \cdot [-0.3, 0.1] + 0.3 = [-0.6, 0.2] + 0.3 = [-0.3, 0.5].$$

This interval contains both positive and negative values, so the function is not monotonic.

Subproblem 2.1: using centered form. Since the function is not monotonic, we have to use the centered form, and compute the following enclosure for the range:

$$[\underline{Y}, \overline{Y}] = \widetilde{y} + \frac{\partial f}{\partial x_1}([\underline{x}_1, \overline{x}_1]) \cdot [-\Delta_1, \Delta_1].$$

Thus,

$$[\underline{Y}, \overline{Y}] = 1.07 + [-0.3, 0.5] \cdot [-0.2, 0.2] = 1.07 + [-0.10, 0.10] = [0.97, 1.17].$$

Subproblem 2.2: preliminary computations. In this case, still $\widetilde{x}_1 = -0.1$. Substituting these values into the expression for the function $f^-(x_1)$, we get the result of data processing:

$$\widetilde{y} = f^-(\widetilde{x}_1) = f^+(-0.1) = (-0.1)^2 + 0.1 \cdot (-0.1) + 1.01 = 0.01 - 0.01 + 1.01 = 1.01.$$

Subproblem 2.2: checking monotonicity. Here,

$$\frac{\partial f^+}{\partial x_1} = 2x_1 + 0.1.$$

We use straightforward interval computations to find the range of this derivative:

$$\frac{\partial f}{\partial x_1}([\underline{x}_1, \overline{x}_1]) = \frac{\partial f}{\partial x_1}([-0.3, 0.1]) =$$

$$2 \cdot [-0.3, 0.1] + 0.1 = [-0.6, 0.2] + 0.1 = [-0.5, 0.3].$$

This interval contains both positive and negative values, so the function is not monotonic.

Subproblem 2.2: using centered form. Since the function is not monotonic, we have to use the centered form, and compute the following enclosure for the range:

$$[\underline{Y}, \overline{Y}] = 1.01 + [-0.5, 0.3] \cdot [-0.2, 0.2] = 1.01 + [-0.10, 0.10] = [0.91, 1.11].$$

Resulting enclosure for Subproblem 2. As planned:

- we take the upper bound ($= 1.17$) from the Subproblem 2.1 corresponding to the function $f^+(x_1) = f(x_1, \overline{x}_2)$, and
- we take the upper bound ($= 0.91$) from the Subproblem 2.2 corresponding to the function $f^-(x_1) = f(x_1, \underline{x}_2)$.

So, for Subproblem 2, we get the enclosure $[0.91, 1.17]$.

Resulting enclosure for the original problem. For Subproblem 1, we got the enclosure $[0.80, 1.22]$. For Subproblem 2, we got the enclosure $[0.91, 1.17]$. To get the enclosure for the original problem, we need to take the union of these two intervals, i.e.:

- take the smallest of their lower bounds, and
- take the largest of their upper bounds.

In our case,

$$[0.80, 1.22] \cup [0.91, 1.17] = [\min(0.80, 0.91), \max(1.22, 1.17)] = [0.80, 1.22].$$

Resulting enclosure – summarizing: $[\underline{Y}, \overline{Y}] = [0.80, 1.22]$. It is narrower than what we got without bisection.