Examples of Linearization

1 Problem and Algorithms: Reminder

Main problem. We know:

- an algorithm $f(x_1, \ldots, x_n)$,
- n value $\widetilde{x}_1, \ldots, \widetilde{x}_n$, and
- n values $\Delta_1, \ldots, \Delta_n$.

We want to find the range

$$[\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i] \text{ for all } i\}.$$

Alternative formulation of the problem. We know:

- an algorithm $f(x_1, \ldots, x_n)$, and
- n intervals $[\underline{x}_i, \overline{x}_i]$.

We want to find the range

$$[\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_i \in [\underline{x}_i, \overline{x}_i] \text{ for all } i\}.$$

How these two formulations are related:

- If we know $\widetilde{x}_i$ and Δ_i, then we take $\underline{x}_i = \widetilde{x}_i - \Delta_i$ and $\overline{x}_i = \widetilde{x}_i + \Delta_i$.
- If we know $\underline{x}_i$ and $\overline{x}_i$, then we take $\widetilde{x}_i = \dfrac{\underline{x}_i + \overline{x}_i}{2}$ and $\Delta_i = \dfrac{\overline{x}_i - \underline{x}_i}{2}$.

Why cannot we just use calculus to solve this problem? Even if for each of the variables, we have only 2 options, for n variables, we need to consider 2^n combinations $(x_1, \ldots, x_n)$. For $n = 1000$, testing all these combinations would take longer than the lifetime of the universe.

First linearization algorithm – that uses actual derivatives: general description.

- First, we compute $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $\Delta = \sum\limits_{i=1}^{n} |c_i| \cdot \Delta_i$, where
$$c_i = \frac{\partial f}{\partial x_i}(\widetilde{x}_1, \ldots, \widetilde{x}_n).$$
- Then, we estimate the desired range as $[\widetilde{y} - \Delta, \widetilde{y} + \Delta]$.

First linearization algorithm: case of $n = 1$.

- First, we compute $\widetilde{y} = f(\widetilde{x}_1)$ and $\Delta = |c_1| \cdot \Delta_1$, where $c_1 = \dfrac{\partial f}{\partial x_1}(\widetilde{x}_1)$.
- Then, we estimate the desired range as $[\widetilde{y} - \Delta, \widetilde{y} + \Delta]$.

First linearization algorithm: case of $n = 2$.

- First, we compute $\widetilde{y} = f(\widetilde{x}_1, \widetilde{x}_2)$ and $\Delta = |c_1| \cdot \Delta_1 + |c_2| \cdot \Delta_2$, where
$$c_i = \frac{\partial f}{\partial x_i}(\widetilde{x}_1, \widetilde{x}_2).$$
- Then, we estimate the desired range as $[\widetilde{y} - \Delta, \widetilde{y} + \Delta]$.

Second linearization algorithm – that uses numerical differentiation: general description. The desired range is estimated . as $[\widetilde{y} - \Delta, \widetilde{y} + \Delta]$, where $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and

$$\Delta = \sum_{i=1}^{n} |f(\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, \widetilde{x}_i + \Delta_i, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n) - \widetilde{y}|.$$

Second linearization algorithm – that uses numerical differentiation: case of $n = 1$. The desired range is est8mated as $[\widetilde{y} - \Delta, \widetilde{y} + \Delta]$, where $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and

$$\Delta = |f(\widetilde{x}_1 + \Delta_1) - \widetilde{y}|$$

Second linearization algorithm – that uses numerical differentiation: case of $n = 2$. The desired range is estimated as $[\widetilde{y} - \Delta, \widetilde{y} + \Delta]$, where $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and

$$\Delta = |f(\widetilde{x}_1 + \Delta_1, \widetilde{x}_2) - \widetilde{y}| + |f(\widetilde{x}_1, \widetilde{x}_2 + \Delta_2) - \widetilde{y}|.$$

2 First Example

Task. Linearize the expression $f(x_1) = x_1^2 + 2$ around the midpoint of the interval $[0, 2]$. Use the linearized expression to find the approximate value of the range of the original function, both with the actual derivative and with the result of numerical differentiation.

Solution: preliminary computations. The midpoint of the interval $[0, 2]$ is the point $\widetilde{x}_1 = \dfrac{0+2}{2} = 1$. For this value, we get

$$\widetilde{y} = f(\widetilde{x}_1) = 1^2 + 2 = 3.$$

The half-width of the interval is equal to $\Delta_1 = \dfrac{2-0}{2} = 1$.

Case when we consider the actual derivative. Here,

$$f'(\widetilde{x}_1) = 2\widetilde{x}_1 = 2 \cdot 1 = 2.$$

Thus,

$$\Delta = |f'(\widetilde{x}_1)| \cdot \Delta_1 = |2| \cdot 1 = 2 \cdot 1 = 2.$$

So, the estimated range is

$$[\widetilde{y} - \Delta, \widetilde{y} + \Delta] = [3 - 2, 3 + 2] = [1, 5].$$

Case when we consider numerical differentiation. In this case,

$$\Delta = |f(\widetilde{x}_1 + \Delta_1) - f(\widetilde{x}_1)|.$$

We have found that $f(\widetilde{x}_1) = 3$. Here,

$$f(\widetilde{x}_1 + \Delta_1) = f(1 + 1) = f(2) = 2^2 + 2 = 6.$$

So, $\Delta = |6 - 3| = 3$, and thus, the estimated range is

$$[\widetilde{y} - \Delta, \widetilde{y} + \Delta] = [3 - 3, 3 + 3] = [0, 6].$$

3 Second Example

Task. Linearize the expression $f(x_1, x_2) = x_1 \cdot x_2 + 2$ for $x_1 \in [0, 2]$ and $x_2 \in [2, 4]$ around the intervals' midpoints. Use the linearized expression to find the approximate value of the range of the original function, both with the actual derivative and with the result of numerical differentiation.

Solution: preliminary computations. The midpoint of the interval $[0, 2]$ is the point $\widetilde{x}_1 = \dfrac{0+2}{2} = 1$, the midpoint of the interval $[2, 4]$ is

$$\widetilde{x}_2 = \frac{2+4}{2} = 3.$$

For these values, we get

$$\widetilde{y} = f(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_1 \cdot \widetilde{x}_2 + 2 = 1 \cdot 3 + 2 = 5.$$

The half-width of the first interval is equal to $\Delta_1 = \dfrac{2-0}{2} = 1$, the half-width of the second interval is equal to $\Delta_2 = \dfrac{4-2}{2} = 1$.

Case when we consider the actual derivatives. Here,

$$c_1 = \frac{\partial f}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_2 = 3,$$

and

$$c_2 = \frac{\partial f}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_1 = 1.$$

Thus,

$$\Delta = |c_1| \cdot \Delta_1 + |c_2| \cdot \Delta_2 = |3| \cdot 1 + |1| \cdot 1 = 3 + 1 = 4.$$

Thus, the estimated range is

$$[\widetilde{y} - \Delta, \widetilde{y} + \Delta] = [5 - 4, 5 + 4] = [1, 9].$$

Case when we consider numerical differentiation. In this case,

$$\Delta = |f(\widetilde{x}_1 + \Delta_1, \widetilde{x}_2) - \widetilde{y}| + |f(\widetilde{x}_1, \widetilde{x}_2 + \Delta_2) - \widetilde{y}| =$$

$$|f(2,3) - 5| + |f(1,4) - 5|.$$

Here,

$$f(2,3) = 2 \cdot 3 + 2 = 6 + 2 = 8,$$

and

$$f(1,4) = 1 \cdot 4 + 2 = 4 + 2 = 6.$$

Thus,

$$\Delta = |8 - 5| + |6 - 5| = 3 + 1 = 4.$$

So, the estimated range is

$$[\widetilde{y} - \Delta, \widetilde{y} + \Delta] = [5 - 4, 5 + 4] = [1, 9].$$