Examples of Using Interval Computations to Locate Maxima

1 Problem and Algorithm: General Reminder

Problem. We want to locate the points at which the given function $f(x_1, \ldots, x_n)$ attains its largest possible value when $x_i \in [\underline{x}_i, \overline{x}_i]$ for all i (i.e., when the tuple $x = (x_1, \ldots, x_n)$ belongs to the box $[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$).

How we solve it. To solve this problem, we use interval computations to estimate the range of the given function on the boxes obtained from the original box by consequent bisections.

In each estimation, we compute the values of the function at some point:

- if the function is monotonic, we compute its values at endpoints, and
- if the function is not monotonic with respect to any of the variables, we use the centered form and thus, compute the value of the function at the midpoints $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$.

All the time, we keep track of the largest-so-far L of all the computed values of the function. At each stage:

- If for our enclosure $[\underline{Y}, \overline{Y}]$ for the range of the function on some box B, we have $\overline{Y} < L$, this means that for all the values $f(x_1, \ldots, x_n)$ for $(x_1, \ldots, x_n) \in B$, we have $f(x_1, \ldots, x_n) \le \overline{y} \le \overline{Y} < L$ thus $f(x_1, \ldots, x_n) < L$. This means that maximum cannot be attained on the box B, so the box B can be dismissed.
- If the function $f(x_1, \ldots, x_n)$ is strictly increasing with respect to some variable x_i, i.e., if the lower bound for the corresponding partial derivative $\dfrac{\partial f}{\partial x_i}$ is positive, this means that the maximum can only be attained if x_i attains its largest possible value $x_i = \overline{x}_i$. In this case, the rest of the box is dismissed, i.e., we replace the original box

  $$[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_{i-1}, \overline{x}_{i-1}] \times [\underline{x}_i, \overline{x}_i] \times [\underline{x}_{i+1}, \overline{x}_{i+1}] \times \ldots \times [\underline{x}_n, \overline{x}_n]$$

  with the smaller box

  $$[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_{i-1}, \overline{x}_{i-1}] \times [\overline{x}_i, \overline{x}_i] \times [\underline{x}_{i+1}, \overline{x}_{i+1}] \times \ldots \times [\underline{x}_n, \overline{x}_n].$$

- Similarly, if the function $f(x_1, \ldots, x_n)$ is strictly decreasing with respect to some variable x_i, i.e., if the upper bound for the corresponding partial derivative $\dfrac{\partial f}{\partial x_i}$ is negative, this means that the maximum can only be attained if x_i attains its smallest possible value $x_i = \underline{x}_i$. In this case, the rest of the box is dismissed, i.e., we replace the original box

$$[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_{i-1}, \overline{x}_{i-1}] \times [\underline{x}_i, \overline{x}_i] \times [\underline{x}_{i+1}, \overline{x}_{i+1}] \times \ldots \times [\underline{x}_n, \overline{x}_n]$$

with the smaller box

$$[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_{i-1}, \overline{x}_{i-1}] \times [\underline{x}_i, \underline{x}_i] \times [\underline{x}_{i+1}, \overline{x}_{i+1}] \times \ldots \times [\underline{x}_n, \overline{x}_n].$$

Comment. Similarly, we can find the points x where the given function attains its smallest possible value. In this case, we keep track of the smallest-so-far value S, and dismiss a box if the lower bound $\underline{Y}$ on this box is larger than the smallest-so-far value S.

2 First Example

Problem. Let us find the values where the function $f(x) = x \cdot (1 - x)$ attains its largest possible value on the interval $[0, 1]$.

What we will do. Strictly speaking, we should use bisection. Bisection is easy to a computer, since computers use binary numbers for which division by 2 simply means adding one more 0 after the decimal point. For us, since we are using decimal notations, it is easier to divide by 5. So, let us divide the interval $[0, 1]$ into five subintervals $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, and $[0.8, 1]$.

The derivative $f'(x)$ is equal to

$$f'(x) = 1 \cdot (1 - x) + x \cdot (-1) = 1 - 2x.$$

First stage: interval $[0, 0.2]$. On the interval $[0, 0.2]$, the range of the derivative is

$$f'([0, 0.2]) = 1 - 2 \cdot [0, 0.2] = 1 - [0.0.4] = [0.6, 1].$$

The lower bound 0.6 of this range is positive. This means that the function $f(x)$ is strictly increasing on this interval. Thus, the largest possible value of $f(x)$ can only be obtained when x is equal to the largest value $x = 0.2$ from this interval. This largest value of $f(x)$ is equal to

$$f(0.2) = 0.2 \cdot (1 - 0.2) = 0.2 \cdot 0.8 = 0.16.$$

The rest of the interval $[0, 0.2]$ can be dismissed. So, instead of the interval $[0, 0.2]$, we consider only the point $x = 0.2$.

At this moment, we have only computed one value of the function $f(x)$, the value $f(0.2) = 0.16$. So, the largest so far value is now $L = 0.16$.

First stage: interval $[0.2, 0.4]$**.** Similarly, we can see that the function $f(x)$ is strictly increasing on the interval $[0.2, 0.4]$, so we keep only the largest value $x = 0.4$ from this interval. For this value,

$$f(0.4) = 0.4 \cdot (1 - 0.4) = 0.4 \cdot 0.6 = 0.24.$$

This value is larger than the previous largest-so-far value 0.16, so the new largest-so-far value is $L = 0.24$.

Now, the largest value 0.16 on the box $[0, 0.2]$ is smaller than this value L. Thus, the whole subbox $[0, 0.2]$ is dismissed. So, out of both subintervals $[0, 0.2]$ and $[0.2, 0.4]$, we only keep the value 0.4.

First stage: interval $[0, 4, 0.6]$**.** On the interval $[0.4, 0.6]$, there is no monotonicity,

$$f'([0.4, 0.6]) = 1 - 2 \cdot [0.4, 0.6] = 1 - [0.8, 1.2] = [-0.2, 0.2].$$

So, we compute $\widetilde{x} = 0.5$,

$$\widetilde{y} = f(\widetilde{x}) = f(0.5) = 0.5 \cdot (1 - 0.5) = 0.5 \cdot 0.5 = 0.25,$$

and $\Delta = 0.1$. The newly computed value $f(0.5) = 0.25)$ is larger than the previous largest-so-far value 0.24, so the new largest-so-far is $L = 0.25$. This means that we can now dismiss the interval $[0.2, 0.4]$ as well, since on this interval, the largest value was $0.24 < L = 0/25$.

To find the range of $f(x)$ on the interval $[0.4, 0.6]$, we use the centered form

$$[\underline{Y}, \overline{Y}] = f(0.5) + f'([0.4, 0.6]) \cdot [-0.1, 0.1] = 0.25 + [-0.2, 0.2] \cdot [-0.1, 0.1] =$$

$$0.25 + [-0.02.0.02] = [0.23, 0.27].$$

The upper endpoint $\overline{Y} = 0.27$ of this range is not smaller than $L = 0.25$, so this interval cannot be dismissed.

First stage: interval $[0.6, 0.8]$**.** On the interval $[0.6, 0.8]$, the function is strictly decreasing. So, the largest value of the function $f(x)$ on this interval is

$$f(0.6) = 0.6 \cdot (1 - 0.6) = 0.6 \cdot 0.4 = 0.24.$$

This value is smaller than the largest-so-far value 0.25, so this interval is dismissed.

First stage: interval $[0.8, 1]$**.** Similarly, on the interval $[0.8, 1]$, the function is strictly decreasing. So, the largest value of the function $f(x)$ on this interval is

$$f(0.8) = 0.8 \cdot (1 - 0.8) = 0.8 \cdot 0.2 = 0.16.$$

This value is smaller than the largest-so-far value 0.25, so this interval is also dismissed.

First stage: conclusion. So, out of all 5 original intervals, only the interval $[0.4, 0.6]$ remains as a possible location of the maximum.

So far, we can only conclude that maximum is located on this interval.

Second stage. To get a more accurate location, we bisect the interval $[0.4, 0.6]$ into two intervals $[0.4, 0.5]$ and $[0.4, 0.6]$.

Second stage: interval $[0.4, 0.5]$. On the first interval $[0.4, 0.5]$, the range of the derivative is

$$f'([0.4, 0.5]) = 1 - 2 \cdot [0.4, 0.5] = 1 - [0.8, 1] = [0, 0.2].$$

The lower bound is non-negative but not positive. So, on this interval, the function is increasing but not necessarily strictly increasing. The upper bound of the function $f(x)$ on this interval is $f(0.5) = 0.25$. This value is not smaller than $L = 0.25$, so this interval cannot be dismissed.

Second stage: interval $[0.5, 0.6]$. Similarly, on the second interval $[0.5, 0.6]$, the range of the derivative is

$$f'([0.5, 0.6]) = 1 - 2 \cdot [0.5, 0.6] = 1 - [1, 1.2] = [-0.2, 0].$$

The upper bound is non-positive but not negative. So, on this interval, the function is decreasing but not necessarily strictly decreasing. The upper bound of the function $f(x)$ on this interval is $f(0.5) = 0.25$. This value is not smaller than $L = 0.25$, so this interval cannot be dismissed.

Second stage: conclusion. So, after the second stage, we keep both intervals $[0.4, 0.5]$ and $[0.5, 0.6]$ as possible locations of the maximum. Thus, we conclude that the maximum is located in one of these intervals, i.e., in their union

$$[0.4, 0.5] \cup [0.5, 0.6] = [0.4, 0.6].$$

Third stage. Let us bisect each of these intervals once again:

- the interval $[0.4, 0.5]$ is divided into two subintervals $[0.4, 0.45]$ and $[0.45, 0.5]$, and
- the interval $[0.5, 0.6]$ is divided into two subintervals $[0.5, 0.55]$ and $[0.55, 0.6]$.

Third stage: interval $[0.4, 0.45]$. On the first interval $[0.4, 0.45]$, the function is increasing, its largest value is

$$f(0.45) = 0.45 \cdot (1 - 0.45) = 0.45 \cdot 0.55 = 0.2425$$

which is smaller than the largest-so-far $L = 0.25$. So, this interval is dismissed.

Third stage: interval $[0.45, 0.5]$**.** On the interval $[0.45, 0.5]$, the range of the derivative is

$$f'([0.45, 0.5]) = 1 - 2 \cdot [0.45, 0.5] = 1 - [0.9, 1] = [0, 0.1].$$

The lower bound is non-negative but not positive. So, on this interval, the function is increasing but not necessarily strictly increasing. The upper bound of the function $f(x)$ on this interval is $f(0.5) = 0.25$. This value is not smaller than $L = 0.25$, so this interval cannot be dismissed.

Third stage: interval $[0.5, 0.55]$**.** On the interval $[0.5, 0.55]$, the range of the derivative is

$$f'([0.5, 0.55]) = 1 - 2 \cdot [0.5, 0.55] = 1 - [1, 1.1] = [-0.1, 0].$$

The upper bound is non-positive but not negative. So, on this interval, the function is decreasing but not necessarily strictly decreasing. The upper bound of the function $f(x)$ on this interval is $f(0.5) = 0.25$. This value is not smaller than $L = 0.25$, so this interval cannot be dismissed.

Third stage: interval $[0.55, 0.6]$**.** On the first interval $[0.55, 0.6]$, the function is decreasing, its largest value is

$$f(0.55) = 0.55 \cdot (1 - 0.55) = 0.55 \cdot 0.45 = 0.2425$$

which is smaller than the largest-so-far $L = 0.25$. So, this interval is dismissed.

Third stage: conclusion. So, after the third stage, we keep intervals $[0.45, 0.5]$ and $[0.5, 0.55]$ as possible locations of the maximum. Thus, we conclude that the maximum is located in one of these intervals, i.e., in their union

$$[0.45, 0.5] \cup [0.5, 0.55] = [0.45, 0.55].$$

Comment. If we want more accurate location, we can further bisect the intervals.

3 Second Example

Problem. Let us find the values where the function $f(x) = x \cdot (1 - x)$ attains its *smallest* possible value on the interval $[0, 1]$.

What we will do. Similarly to the first example, let us divide the interval $[0, 1]$ into five subintervals $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, and $[0.8, 1]$.

First stage: interval $[0, 0.2]$**.** On the interval $[0, 0.2]$, the function $f(x)$ is strictly increasing, so its minimum can be attained only at the smallest possible value $x = 0$. For this value, we have

$$f(0) = 0 \cdot (1 - 0) = 0,$$

Thus, out of this interval, we only keep the point $x = 0$, and the smallest-so-far-value is $S = 0$.

First stage: interval $[0.2, 0.4]$**.** On the interval $[0.2, 0.4]$, the function is also strictly increasing, so its smallest possible value is $f(0.2) = 0.16$. This value is larger than the smallest-so-far $S = 0$, so this interval is dismissed.

First stage: interval $[0.4, 0.6]$**.** On the interval $[0.4, 0.6]$, the function is not monotonic. Centered form leads to $\underline{Y} = 0.23$. This value is larger than the smallest-so-far $S = 0$, so this interval is dismissed.

First stage: interval $[0.6, 0.8]$**.** On the interval $[0.6, 0.8]$, the function is strictly decreasing, so its smallest possible value is $f(0.8) = 0.16$. This value is larger than the smallest-so-far $S = 0$, so this interval is dismissed.

First stage: interval $[0.8, 1]$**.** On the interval $[0.8, 1.0]$, the function is also strictly decreasing, so its minimum can be attained only at the largest possible value $x = 1$. For this value, we have

$$f(1) = 1 \cdot (1 - 1) = 1 \cdot 0 = 0.$$

Thus, out of this interval, we only keep the point $x = 1$.

First stage: conclusion. We conclude that the minimum is attained at two points: $x = 0$ and $x = 1$.

4 Third example

Problem. Let us locate the maxima of the function $f(x_1, x_2) = x_1 \cdot x_2$ when $x_1 \in [-3, 1]$ and $x_2 \in [-1, 3]$.

First bisection. Let us first decide which variable to use for bisection. In this example, $\widetilde{x}_1 = -1$, $\widetilde{x}_2 = 1$, $\Delta_1 = \Delta_2 = 2$,

$$\frac{\partial f}{\partial x_1} = x_2, \quad \frac{\partial f}{\partial x_2} = x_1,$$

so

$$c_1 = \frac{\partial f}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_2 = 1, \quad c_2 = \frac{\partial f}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_1 = -1.$$

Thus, $|c_1| \cdot \Delta_1 = 1 \cdot 2 = 2$ and $|c_2| \cdot \Delta_1 = 1 \cdot 2 = 2$. These values are equal, so we bisect by x_1, into two intervals $x_1 \in [-3, -1]$ and $x_1 \in [-1, 1]$. So, we get two subboxes: $[-3, -1] \times [-1, 3]$ and $[-1, 1] \times [-1, 3]$.

First stage: box $B = [-3, -1] \times [-1, 3]$**.** On this subbox B, we have

$$\frac{\partial f}{\partial x_1}(B) = [\underline{x}_2, \overline{x}_2] = [-1, 3].$$

This interval includes both positive and negative values, so the function is not monotonic with respect to x_1.

Let us check monotonicity with respect to x_2. Here,

$$\frac{\partial f}{\partial x_2}(B) = [\underline{x}_1, \overline{x}_1] = [-3, -1].$$

The upper bound is negative, so $f(x_1, x_2)$ is strictly decreasing with respect to x_2. Thus, the largest possible value can only be attained when x_2 is the smallest, i.e., when $x_2 = -1$, all other points from this subbox can be safely dismissed.

For $x_2 = -1$, the function $f(x_1, x_2)$ takes the form $f^+(x_1) = (-1) \cdot x_1$. This function is strictly decreasing, so its largest possible value is attained when x_1 is the smallest, i.e., when $x_1 = -3$. For $x_1 = -3$ and $x_2 = -1$, we have $f(x_1, x_2) = (-3) \cdot (-1) = 3$. All other pairs (x_1, x_2) from this box can be dismissed. Out of the whole box $[-3, -1] \times [-1, 3]$, we only keep the point $(-3, -1)$ as a possible location of the maximum.

This is the first value of the function $f(x_1, x_2)$ that we are computing, so the largest-so-far value is $L = 3$.

First stage: second box $B = [-1, 1] \times [-1, 3]$. On this subbox,

$$\frac{\partial f}{\partial x_1}(B) = [\underline{x}_2, \overline{x}_2] = [-1, 3]$$

and

$$\frac{\partial f}{\partial x_2}(B) = [\underline{x}_1, \overline{x}_1] = [-1, 1].$$

This function is not monotonic with respect to any of the variables, so to find its range, we use the centered form. Here, $\widetilde{x}_1 = 0$, $\widetilde{x}_2 = 1$, $\Delta_1 = 1$, $\Delta_2 = 2$, and $\widetilde{y} = f(\widetilde{x}_1, \widetilde{x}_2) = 0 \cdot 1 = 0$, so the estimate is

$$[\underline{Y}, \overline{Y}] = 0 + [-1, 3] \cdot [-1, 1] + [-1, 1] \cdot [-2, 2] = [-3, 3] + [-2, 2] = [-5, 5].$$

The upper bound $\overline{Y} = 5$ is not smaller than the largest-so-far value $L = 3$, so this interval cannot be dismissed.

Conclusion of the first stage. The maximum is attained either at a point $(-3, -1)$, or in the box $[-1, 1] \times [-1, 3]$. To get a more accurate location, we need to bisect this box.

Second bisection. Let us decide which variable to use for bisection. In this example, $\widetilde{x}_1 = 0$, $\widetilde{x}_2 = 1$, $\Delta_1 = 1$, $\Delta_2 = 2$, so

$$c_1 = \frac{\partial f}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_2 = 1, \quad c_2 = \frac{\partial f}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_1 = 0.$$

Thus, $|c_1| \cdot \Delta_1 = 1 \cdot 1 = 1$ and $|c_2| \cdot \Delta_1 = 0 \cdot 2 = 0$. The first product is larger, so we bisect by x_1 into two intervals $x_1 \in [-1, 0]$ and $x_1 \in [0, -1]$. So, we get two subboxes: $[-1, 0] \times [-1, 3]$ and $[0, 1] \times [-1, 3]$.

Second stage: box $B = [-1, 0] \times [-1, 3]$**.** On this subbox B, we have

$$\frac{\partial f}{\partial x_1}(B) = [\underline{x}_2, \overline{x}_2] = [-1, 3].$$

This interval includes both positive and negative values, so the function is not monotonic with respect to x_1.

Let us check monotonicity with respect to x_2. Here,

$$\frac{\partial f}{\partial x_2}(B) = [\underline{x}_1, \overline{x}_1] = [-1, 0].$$

The upper bound is non-positive, so $f(x_1, x_2)$ is decreasing (not necessarily strictly) with respect to x_2. Thus, the largest possible value is attained when x_2 is the smallest, i.e., when $x_2 = -1$ – but we cannot dismiss anything. For $x_2 = -1$, as we have already found out, the function is decreasing with respect to x_1, so its largest value is attained when x_1 is the smallest, i.e., when $x_1 = -1$. Thus, the largest possible value of the function on this box is attained when $x_1 = x_2 = -1$, this value is $f(x_1, x_2) = (-1) \cdot (-1) = 1$. It is smaller than the largest-so-far $L = 3$, so this box is dismissed.

Second stage: box $B = [0, 1] \times [-1, 3]$**.** On this subbox B, we have

$$\frac{\partial f}{\partial x_1}(B) = [\underline{x}_2, \overline{x}_2] = [-1, 3].$$

This interval includes both positive and negative values, so the function is not monotonic with respect to x_1.

Let us check monotonicity with respect to x_2. Here,

$$\frac{\partial f}{\partial x_2}(B) = [\underline{x}_1, \overline{x}_1] = [0, 1].$$

The lower bound is non-negative, so $f(x_1, x_2)$ is increasing (not necessarily strictly) with respect to x_2. Thus, the largest possible value is attained when x_2 is the largest, i.e., when $x_2 = 3$ – but we cannot dismiss anything. For $x_2 = 3$, we get $f^+(x_1) = 3x_1$. This function is increasing with respect to x_1, so its largest value is attained when x_1 is the largest, i.e., when $x_1 = 1$. Thus, the largest possible value of the function on this box is attained when $x_1 = 1$ and $x_2 = 3$, this value is $f(x_1, x_2) = 1 \cdot 3 = 3$. This value is not smaller than the largest-so-far $L = 3$, so this box cannot be dismissed.

Conclusion of the second stage. The maximum is attained either at a point $(-3, -1)$, or in the box $[0, 1] \times [-1, 3]$. To get a more accurate location, we need to bisect this box.

Third bisection. Let us decide which variable to use for bisection. In this example, $\widetilde{x}_1 = 0.5$, $\widetilde{x}_2 = 1$, $\Delta_1 = 0.5$, $\Delta_2 = 2$, so

$$c_1 = \frac{\partial f}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_2 = 1, \quad c_2 = \frac{\partial f}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_1 = 0.5.$$

Thus, $|c_1| \cdot \Delta_1 = 1 \cdot 0.5 = 0.5$ and $|c_2| \cdot \Delta_1 = 0.5 \cdot 2 = 1$. The second product is larger, so we bisect by x_2 into two intervals $x_2 \in [-1, 1]$ and $x_2 \in [1, 3]$. So, we get two subboxes: $[0, 1] \times [-1, 1]$ and $[0, 1] \times [1, 3]$.

Third stage: box $B = [0, 1] \times [-1, 1]$. On this subbox B, we have

$$\frac{\partial f}{\partial x_1}(B) = [\underline{x}_2, \overline{x}_2] = [-1, 1].$$

This interval includes both positive and negative values, so the function is not monotonic with respect to x_1.

Let us check monotonicity with respect to x_2. Here,

$$\frac{\partial f}{\partial x_2}(B) = [\underline{x}_1, \overline{x}_1] = [0, 1].$$

The upper bound is non-negative, so $f(x_1, x_2)$ is increasing (not necessarily strictly) with respect to x_2. Thus, the largest possible value is attained when x_2 is the largest, i.e., when $x_2 = 1$ – but we cannot dismiss anything. For $x_2 = 1$, we get $f^+(x_1) = x_1 \cdot 1 = x_1$. This function is increasing with respect to x_1, so its largest value is attained when x_1 is the largest, i.e., when $x_1 = 1$. Thus, the largest possible value of the function on this box is attained when $x_1 = 1$ and $x_2 = 1$, this value is $f(x_1, x_2) = 1 \cdot 1 = 1$. It is smaller than the largest-so-far $L = 3$, so this box is dismissed.

Third stage: box $B = [0, 1] \times [1, 3]$. On this subbox B, we have

$$\frac{\partial f}{\partial x_1}(B) = [\underline{x}_2, \overline{x}_2] = [1, 3].$$

The lower bound is positive, thus the function is strictly increasing with respect to x_1. So, the largest value can only be attained when x_1 is the largest, i.e., when $x_1 = 1$. All other values can be dismissed.

For $x_1 = 1$, the function takes the form $f(x_1, x_2) = x_2$. This function is strictly increasing with respect to x_2, so the maximum can only be attained when x_2 attains its largest possible value $x_2 = 3$, all other pairs can be dismissed. For $x_1 = 1$ and $x_2 = 3$, we get $f(x_1, x_2) = 3$. This is exactly equal to the largest-so-far value $L = 3$.

Conclusion. The maximum is attained either at the point $(x_1, x_2) = (-3, -1)$ or at the point $(x_1, x_2) = (1, 3)$.