How to Use Monotonicity in Interval Computations: A Detailed Description

Main problem of interval computations: reminder. We are given:

- an *algorithm* $y = f(x_1, \ldots, x_n)$ that inputs n real numbers $x_1, \ldots, x_n$ and returns the value $y = f(x_1, \ldots, x_n)$, and
- n *intervals* $[\underline{x}_1, \overline{x}_1]$, ..., $[\underline{x}_n, \overline{x}_n]$.

We want to compute the range of f on these intervals, i.e., the set

$$\left[\underline{y}, \overline{y}\right] = f([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \stackrel{\text{def}}{=}$$

$$\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}$$

or at least an *enclosure* for this range, i.e., the interval $\left[\underline{Y}, \overline{Y}\right]$ for which

$$\left[\underline{y}, \overline{y}\right] \subseteq \left[\underline{Y}, \overline{Y}\right].$$

Relation to optimization. The function $f(x_1, \ldots, x_n)$ is defined on the *box*

$$B \stackrel{\text{def}}{=} [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n],$$

i.e., on the set of all tuples $(x_1, \ldots, x_n)$ for which

$$x_1 \in [\underline{x}_1, \overline{x}_1], \ \ldots, \text{ and } x_n \in [\underline{x}_n, \overline{x}_n].$$

In these terms, the endpoints $\underline{y}$ and $\overline{y}$ are simply the smallest and the largest value of this function on this box:

$$\underline{y} = \min_{x \in B} f(x) \text{ and } \underline{y} = \max_{x \in B} f(x).$$

Straightforward interval computations: reminder. To find an enclosure, we can:

- *parse* the expression $f(x_1, \ldots, x_n)$, i.e., represent it – as a compiler does – as a sequence of elementary arithmetic operations, and then

- replace each operation with numbers by the corresponding operation with intervals.

For example, if we want to compute the range of the function $y = (x-1)^2$ on the interval $X = [0,2]$, we do the following. First, we represent the computation of y as a sequence of elementary operations:

$$r_1 = x - 1;$$

$$y = r_1 \cdot r_1.$$

Then, we replace each of these operations by the corresponding operation of interval arithmetic:

$$R_1 = X - 1 = [0,2] - 1 = [-1,1]\,;$$

$$Y = R_1 \cdot R_1 = [-1,1] \cdot [-1,1] =$$

$$[\min((-1)\cdot(-1), (-1)\cdot 1, 1\cdot(-1), 1\cdot 1), \max((-1)\cdot(-1), (-1)\cdot 1, 1\cdot(-1), 1\cdot 1)] =$$

$$[\min(1,-1,-1,1), \max(1,-1,-1,1)] = [-1,1]\,.$$

Idea of using monotonicity. If for some i, the function $f(x_1,\ldots,x_n)$ is increasing with respect to the variable x_i, then:

- the largest value $\overline{y}$ of the function f on the box B is attained when x_i attains its largest possible value $x_i = \overline{x}_i$, and
- the smallest value $\underline{y}$ of the function f on the box B is attained when x_i attains its smallest possible value $x_i = \underline{x}_i$.

Thus:

- To compute $\overline{y}$, we do not need to compute the range of the function $f(x_1,\ldots,x_n)$ of all n variables over the whole box B, it is sufficient to estimate the range of the function

  $$f_i^+(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = f(x_1,\ldots,x_{i-1},\overline{x}_i,x_{i+1},\ldots,x_n)$$

  of $n-1$ variables on the box

  $$B_i = [\underline{x}_1,\overline{x}_1] \times \ldots \times \left[\underline{x}_{i-1},\overline{x}_{i-1}\right] \times \left[\underline{x}_{i+1},\overline{x}_{i+1}\right] \times \ldots \times [\underline{x}_n,\overline{x}_n]\,.$$

- To compute $\underline{y}$, we do not need to compute the range of the function $f(x_1,\ldots,x_n)$ of all n variables over the whole box B, it is sufficient to estimate the range of the function

  $$f_i^-(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = f(x_1,\ldots,x_{i-1},\underline{x}_i,x_{i+1},\ldots,x_n)$$

  of $n-1$ variables on the box B_i.

Similarly, if for some i, the function $f(x_1, \ldots, x_n)$ is decreasing with respect to the variable x_i, then:

- the largest value $\overline{y}$ of the function f on the box B is attained when x_i attains its smallest possible value $x_i = \underline{x}_i$, and
- the smallest value $\underline{y}$ of the function f on the box B is attained when x_i attains its largest possible value $x_i = \overline{x}_i$.

Thus:

- To compute $\overline{y}$, we do not need to compute the range of the function $f(x_1, \ldots, x_n)$ of all n variables over the whole box B, it is sufficient to estimate the range of the function $f_i^-(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ of $n-1$ variables on the box B_i.
- To compute $\underline{y}$, we do not need to compute the range of the function $f(x_1, \ldots, x_n)$ of all n variables over the whole box B, it is sufficient to estimate the range of the function $f_i^+(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ of $n-1$ variables on the box B_i.

How do we know when the function is monotonic? According to calculus, a function f is increasing with respect to the variable x_i if its derivative $\dfrac{\partial f}{\partial x_i}$ is non-negative. Thus, to check whether the function f is increasing with respect to x_i for all points on the box, it is sufficient to find the range $[\underline{r}_i, \overline{r}_i]$ of the partial derivative on this box. If $\underline{r}_i \geq 0$, this means that all the values of the partial derivative on this box are non-negative, so the function is increasing.

In practice, we often cannot compute the exact range, we can only compute the enclosure $\left[\underline{R}_i, \overline{R}_i\right] \supseteq [\underline{r}_i, \overline{r}_i]$ for this range. Then, if $\underline{R}_i \geq 0$, we conclude that $\underline{r}_i \geq \underline{R}_i \geq 0$ and thus, that the function f is increasing with respect to x_i.

Similarly, according to calculus, a function f is decreasing with respect to the variable x_i if its derivative $\dfrac{\partial f}{\partial x_i}$ is non-positive. Thus, to check whether the function f is decreasing with respect to x_i for all points on the box, it is sufficient to find the range $[\underline{r}_i, \overline{r}_i]$ of the partial derivative on this box. If $\overline{r}_i \leq 0$, this means that all the values of the partial derivative on this box are non-positive, so the function is decreasing.

In practice, we often cannot compute the exact range, we can only compute the enclosure $\left[\underline{R}_i, \overline{R}_i\right] \supseteq [\underline{r}_i, \overline{r}_i]$ for this range. Then, if $\overline{R}_i \leq 0$, we conclude that $\overline{r}_i \leq \overline{R}_i \leq 0$ and thus, that the function f is decreasing with respect to x_i.

Resulting (recursive) monotonicity-using algorithm M. In order to use monotonicity when solving the above-mentioned main problem of interval computations, we do the following. For $i = 1, 2, \ldots$, we use straightforward interval computations to find an enclosure $\left[\underline{R}_i, \overline{R}_i\right]$ for the range of the i-th partial derivative on the box B.

If for some i, this enclosure does not imply monotonicity – i.e., if $\underline{R}_i < 0$ and $\overline{R}_i > 0$, then we go to the next i. If for none of the variables i, we have monotonicity, this means that to find the desired enclosure $\left[\underline{Y}, \overline{Y}\right]$, we need to use straightforward interval computations – or some other method.

If for some i, we have $\underline{R}_i \geq 0$, then we use the same algorithm M to solve the following two interval computation problems with fewer inputs:

- first, we find the enclosure $\left[\underline{Y}^+, \overline{Y}^+\right]$ for the problem of estimating the range of the function f^+ on the box B_i;
- then, we find the enclosure $\left[\underline{Y}^-, \overline{Y}^-\right]$ for the problem of estimating the range of the function f^- on the box B_i;
- finally, we take $\left[\underline{Y}^-, \overline{Y}^+\right]$ as the enclosure for the original problem.

If for some i, we have $\overline{R}_i \geq 0$, then we use the same algorithm M for solve the following two interval computation problems with fewer inputs:

1. first, we find the enclosure $\left[\underline{Y}^+, \overline{Y}^+\right]$ for the problem of estimating the range of the function f_i^+ on the box B_i;
2. then, we find the enclosure $\left[\underline{Y}^-, \overline{Y}^-\right]$ for the problem of estimating the range of the function f_i^- on the box B_i;
3. finally, we take $\left[\underline{Y}^+, \overline{Y}^-\right]$ as the enclosure for the original problem.

Example. Let us estimate the range of the function $f(x_1, x_2) = (x_1^2+1)\cdot(x_2-1)$ when $x_1 \in [-1, 1]$ and $x_2 \in [0, 2]$.

In this case,
$$\frac{\partial f}{\partial x_1} = 2x_1 \cdot (x_2 - 1).$$
Here, straightforward interval computations lead to
$$\left[\underline{R}_1, \overline{R}_1\right] = 2 \cdot [-1, 1] \cdot ([0, 2] - [1, 1]) = [-2, 2] \cdot [-1, 1] = [-2, 2].$$
Here, $\underline{R}_1 < 0 < \overline{R}_1$, so we cannot make any conclusion whether the given function f is monotonic with respect to x_1. Thus, we move to the next value i, i.e., to the value $i = 2$.

For $i = 2$, we have
$$\frac{\partial f}{\partial x_2} = x_1^2 + 1.$$
Here, straightforward interval computations lead to
$$\left[\underline{R}_2, \overline{R}_2\right] = [-1, 1] \cdot [-1, 1] + 1 = [-1, 1] + 1 = [0, 2].$$

Here, $\underline{R}_2 \geq 0$. So, instead of the original problem of finding the range of the function f of two variables, we have to solve two problems of finding the range of the two functions

$$f_2^+(x_1) = (x_1^2+1)\cdot(\overline{x}_2 - 1) = (x_1^2+1)\cdot(2-1) = x_1^2+1$$

and

$$f_2^-(x_1) = (x_1^2+1)\cdot(\underline{x}_2 - 1) = (x_1^2+1)\cdot(0-1) = -x_1^2-1$$

of $n-1=1$ variable on the box $B_2 = [-1,1]$.

1. First, we find the enclosure $\left[\underline{Y}^+, \overline{Y}^+\right]$ for the problem of estimating the range of the function $f_2^+(x_1) = x_1^2+1$ on the box $B_2 = [-1,1]$. To find this enclosure, we use the same algorithm M.

Namely, we find the enclosure of the derivative

$$\frac{df_2^+}{dx_1} = 2x_1$$

on the interval $[-1,1]$. This range is estimated as $2\cdot[-1,1] = [-2,2]$. This range includes both positive and negative values. Thus, to find the range $\left[\underline{Y}^+, \overline{Y}^+\right]$ of the function $f_2^+(x_1) = x_1^2+1$, we cannot use monotonicity, we have to use straightforward interval computations. We thus get

$$\left[\underline{Y}^+, \overline{Y}^+\right] = [-1,1]\cdot[-1,1]+1 = [-1,1]+1 = [0,2]\,.$$

2. Then, we find the enclosure $\left[\underline{Y}^-, \overline{Y}^-\right]$ for the problem of estimating the range of the function $f_2^-(x_1) = -x_1^2-1$ on the box $B_2 = [-1,1]$. To find this enclosure, we use the same algorithm M.

Namely, we find the enclosure of the derivative

$$\frac{df_2^-}{dx_1} = -2x_1$$

on the interval $[-1,1]$. This range is estimated as $-2\cdot[-1,1] = [-2,2]$. This range includes both positive and negative values. Thus, to find the range $\left[\underline{Y}^-, \overline{Y}^-\right]$ of the function $f_2^+(x_1) = -x_1^2-1$, we cannot use monotonicity, we have to use straightforward interval computations. We thus get

$$\left[\underline{Y}^-, \overline{Y}^-\right] = -[-1,1]\cdot[-1,1]-1 = [-1,1]-1 = [-2,0]\,.$$

3. Finally, we take $\left[\underline{Y}^-, \overline{Y}^+\right] = [-2,2]$ as the enclosure for the original problem.