How Calculus Helps Find the Range: General Case

Problem: reminder.

- We are given a function $y = f(x_1, \ldots, x_n)$ and ranges $[\underline{x}_i, \overline{x}_i]$, $i = 1, \ldots, n$.
- We want to find the range

$$[\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_i \in [\underline{x}_i, \overline{x}_i] \text{ for all } i\}$$

  of all possible values $y = f(x_1, \ldots, x_n)$ when $x_i \in [\underline{x}_i, \overline{x}_i]$ for all i.

Idea. In this case,

$$\underline{y} = \min_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n); \quad \overline{y} = \max_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n).$$

When a function attains maximum or minimum, then if we only consider its dependence on each of the variables, we only get minimum or maximum. Thus, for each variable i, either $x_i = \underline{x}_i$, or $x_i = \overline{x}_i$, or $\dfrac{\partial f}{\partial x_i} = 0$.

Resulting algorithm. We consider all possible combinations $(x_1, \ldots, x_n)$ for which, for each i, we have:

- either $x_i = \underline{x}_i$,
- or $x_i = \overline{x}_i$,
- or $\dfrac{\partial f}{\partial x_i} = 0$ and $x_i \in [\underline{x}_i, \overline{x}_i]$.

For each such combination, we compute the value $f(x_1, \ldots, x_n)$. The smallest of these values of the function is $\underline{y}$, the largest is $\overline{y}$.

Caution. If for each variable, we have 3 options, this means that we have to consider 3^n possible combinations. For large n, this is not feasible.

Example. Let us compute the range of a function $f(x_1, x_2) = x_1^2 - x_1 \cdot x_2 + x_2^2$ when x_1 is in the interval $[-1, 1]$ and x_2 is in the interval $[-1, 1]$.

Here, the derivative with respect to x_1 is equal to $2x_1 - x_2$. So, for x_1, we need to consider three cases:

- when $x_1 = -1$,
- when $x_1 = 1$, and
- when $\dfrac{\partial f}{\partial x_1} = 0$, i.e., when $2x_1 - x_2 = 0$.

The derivative with respect to x_2 is equal to $-x_1 + 2x_2$. So, for x_2, we need to consider two cases:

- when $x_2 = -1$,
- when $x_2 = 1$, and
- when $\dfrac{\partial f}{\partial x_2} = 0$, i.e., when $-x_1 + 2x_2 = 0$.

First, let us consider cases when $x_1 = -1$.

- When $x_1 = -1$ and $x_2 = -1$, we get
$$f(x_1, x_2) = (-1)^2 - (-1) \cdot (-1) + (-1)^2 = 1 - 1 + 1 = 1.$$
- When $x_1 = -1$ and $x_2 = 1$, we get
$$f(x_1, x_2) = (-1)^2 - (-1) \cdot 1 + 1^2 = 1 + 1 + 1 = 3.$$
- When $x_1 = -1$ and $-x_1 + 2x_2 = 0$, then $x_2 = -0.5$, so
$$f(x_1, x_2) = (-1)^2 - (-1) \cdot (-0.5) + (-0.5)^2 = 1 - 0.5 + 0.25 = 0.75.$$

Second, we consider cases when $x_1 = 1$:

- When $x_1 = 1$ and $x_2 = -1$, we get
$$f(x_1, x_2) = 1^2 - 1 \cdot (-1) + (-1)^2 = 1 + 1 + 1 = 3.$$
- When $x_1 = 1$ and $x_2 = 1$, we get
$$f(x_1, x_2) = 1^2 - 1 \cdot 1 + 1^2 = 1 - 1 + 1 = 1.$$
- When $x_1 = 1$ and $-x_1 + 2x_2 = 0$, then $x_2 = 0.5$, so
$$f(x_1, x_2) = 1^2 - 1 \cdot 0.5 + 0.5^2 = 1 - 0.5 + 0.25 = 0.75.$$

Finally, when $\dfrac{\partial f}{\partial x_1} = 0$, i.e., when $2x_1 - x_2 = 0$, then:

- When $2x_1 - x_2 = 0$ and $x_2 = -1$, we get $x_1 = -0.5$, so
$$f(x_1, x_2) = (-0.5)^2 - (-0.5) \cdot (-1) + (-1)^2 = 0.25 - 0.5 + 1 = 0.75.$$

- When $2x_1 - x_2 = 0$ and $x_2 = 1$, we get $x_1 = 0.5$, so

$$f(x_1, x_2) = 0.5^2 - 0.5 \cdot 1 + 1^2 = 0.25 - 0.5 + 1 = 0.75.$$

- When $2x_1 - x_2 = 0$ and $-x_1 + 2x_2 = 0$, then substituting $x_2 = 2x_1$ from the first equation into the second equation, we get $-x_1 + 4x_1 = 0$, i.e., $3x_1 = 0$ and $x_1 = 0$. Thus, $x_2 - 2x_1 = 0$, and we have

$$f(x_1, x_2) = 0^2 - 0 \cdot 0 + 0^2 = 0 - 0 + 0 = 0.$$

So, we get the following values:

	$x_1 = -1$	$x_1 = 1$	$\dfrac{\partial f}{\partial x_1} = 0$
$x_2 = -1$	1	3	0.75
$x_2 = 1$	3	1	0.75
$\dfrac{\partial f}{\partial x_2} = 0$	0.75	0.75	0

The smallest of these values is 0, the largest is 3, so the range is $[0, 3]$.