Solution to Problem 10

Task. Use monotonicity and one iteration of bisection to estimate the range of the function from Problem 2 when x_1 is in the interval $[0, 6]$ and x_2 is in the interval $[-6, 6]$.

Solution.

Which variable to bisect. Here,

$$\frac{\partial f}{\partial x_1} = -4 - x_2, \quad \frac{\partial f}{\partial x_2} = 2 - x_1.$$

Midpoints and half-widths are:

$$\underline{x}_1 = \frac{0+6}{2} = 3, \quad \Delta_1 = \frac{6-0}{2} = 3, \quad \widetilde{x}_2 = \frac{-6+6}{2} = 0, \quad \Delta_2 = \frac{6-(-6)}{2} = 6.$$

Thus,

$$\frac{\partial f}{\partial x_1}(\widetilde{x}_1, \widetilde{x}_2) = -4 - \widetilde{x}_2 = -4 - 0 = -4, \text{ so}$$

$$\left|\frac{\partial f}{\partial x_1}\right| \cdot \Delta_1 = |-4| \cdot 3 = 4 \cdot 3 = 12;$$

$$\frac{\partial f}{\partial x_2}(\widetilde{x}_1, \widetilde{x}_2) = 2 - \widetilde{x}_1 = 2 - 3 = -1, \text{ so}$$

$$\left|\frac{\partial f}{\partial x_2}\right| \cdot \Delta_2 = |-1| \cdot 6 = 1 \cdot 6 = 6.$$

Here,

$$\left|\frac{\partial f}{\partial x_1}\right| \cdot \Delta_1 > \left|\frac{\partial f}{\partial x_2}\right| \cdot \Delta_2,$$

so we bisect x_1, i.e., replace the original interval $[0, 6]$ for x_1 with two intervals $[0, 3]$ and $[3.6]$.

Case when $x_1 \in [0, 3]$ and $x_2 \in [-6, 6]$. Let us first check monotonicity. The range of the first derivative is

$$-4 - [-6, 6] = [-4, -4] - [-6, 6] = [-4 - 6, -4 - (-6)] = [-10, 2].$$

This range contains both positive and negative values, so on this sub-box the given function is not monotonic with respect to x_1.

The range of the second derivative is

$$2 - [0,3] = [2,2] - [0,3] = [2-3, 2-0] = [-1,2].$$

This range also contains both positive and negative values, so on this sub-box the given function is not monotonic with respect to x_2 either.

Thus, to find the range on this sub-box, we need to use straightforward interval computations:

$$(2 - [0,3]) \cdot (4 + [-6,6]) = [-1,2] \cdot [-2,10] = [-10,20].$$

Case when $x_1 \in [3,6]$ and $x_2 \in [-6,6]$. Let us first check monotonicity. The range of the first derivative is

$$-4 - [-6,6] = [-4,-4] - [-6,6] = [-4-6, -4-(-6)] = [-10,2].$$

This range contains both positive and negative values, so on this sub-box the given function is not monotonic with respect to x_1.

The range of the second derivative is

$$2 - [3,6] = [2,2] - [3,6] = [2-6, 2-3] = [-4,-1].$$

All values from this interval are negative, so on this sub-box the given function is monotonic with respect to x_2. So:

- to find $\underline{y}$, it is sufficient to consider the largest possible value of x_2, namely, $x_2 = 6$; in this case,

$$f(x_1, 6) = (2 - x_1) \cdot (4 + 6) = 10 \cdot (2 - x_1);$$

- to find $\overline{y}$, it is sufficient to consider the smallest possible value of x_2, namely, $x_2 = -6$; in this case,

$$f(x_1, 6) = (2 - x_1) \cdot (4 + (-6)) = -2 \cdot (2 - x_1).$$

Let us check monotonicity of the resulting functions of one variable x_1.

- For the function $10 \cdot (2 - x_1)$ (that we use to estimate $\underline{y}$), the derivative is $-10 < 0$, so this function is decreasing. Thus, its smallest value $\underline{y}$ is attained when x_1 is the largest, i.e., when $x_1 = 6$. For this value x_1, we have

$$\underline{y} = f(x_1, x_2) = 10 \cdot (2 - 6) = 10 \cdot (-4) = -40.$$

- For the function $-2 \cdot (2 - x_1)$ (that we use to estimate $\overline{y}$), the derivative is $2 > 0$, so this function is increasing. Thus, its largest value $\overline{y}$ is attained when x_1 is the largest, i.e., when $x_1 = 6$. For this value x_1, we have

$$\overline{y} = f(x_1, x_2) = -2 \cdot (2 - 6) = -2 \cdot (-4) = 8.$$

So, the range of the given function on this sub-box is estimated as $[-40, 8]$.

Final answer. To find the range over the whole original box, we need to take the union of ranges over two subboxes:

$$[-10, 20] \cup [-40, 8] = [-40, 20].$$