Solution to Problem 14

Task. Use the interval-based optimization algorithm to locate the maximum of the function $f(x) = 2x + 4x^2$ on the interval $[-0.4, 0]$. Divide this interval into two, then divide the remaining intervals into two again, etc. At each iteration, dismiss subintervals where maximum cannot be attained. Stop when you get intervals of width 0.1.

Solution. Let us first divide this interval into two subintervals: $[-0.4, -0.2]$ and $[-0.2, 0]$.

For the first subinterval, the range of the derivative is equal to

$$2 + 8 \cdot [-0.4, -0.2] = 2 + [-3.2, -1.6] = [-1.2, 0.4].$$

This range contains both positive and negative values, so we have to use the centered form. For this subinterval, the midpoint is $\widetilde{x} = -0.3$. At this point, the value of the function is

$$f(-0.3) = 2 \cdot (-0.3) + 4 \cdot (-0.3)^2 = -0.6 + 4 \cdot 0.09 = -0.6 + 0.36 = -0.24.$$

The half-width of this subinterval is $\Delta = 0.1$. Thus, the centered form leads to the following enclosure for the range:

$$f(\widetilde{x}) + f'([\underline{x}, \overline{x}]) \cdot [-\Delta, \Delta] = -0.24 + [-1.2, 0.4] \cdot [-0.1, 0.1] =$$

$$-0.24 + [-0.12, 0.12] = [-0.36, -0.12].$$

On the second interval, the range of the derivative $2 + 8x$ is equal to

$$2 + 8 \cdot [-0.2, 0] = 2 + [-1.6, 0] = [0.4, 2].$$

All these values are positive, so the function is increasing on this subinterval.

- Its smallest value is attained when x is the smallest $x = -0.2$ and is equal to

  $$f(-0.2) = 2 \cdot (-0.2) + 4 \cdot (-0.2)^2 = -0.4 + 4 \cdot 0.04 = -0.4 + 0.16 = -0.24.$$

- Its largest value is attained when x is the largest $x = 0$ and is equal to $f(0) = 0$.

Thus, the range of the function on the first subinterval is $[-0.24, 0]$.

We have computed the value $f(x)$ for three values x: $x = -0.3$, $x = -0.2$, and $x = 0$. The largest if the resulting three values is $f(0) = 0$. All the values $f(x)$ for x from the first subinterval are smaller than or equal to -0.12, which is smaller than 0. Thus, the maximum cannot be attained on the first subinterval, so it is attained on the second subinterval.

Since on the second subinterval, the function is increasing, its maximum there is attained at the smallest value $x = 0$. Thus, the maximum of the function $f(x)$ on the whole interval $[-0.4, 0]$ is attained when $x = 0$.