Solution to Problem 24

**Task.** Let us assume that:

- the quantity $x$ is described by the membership function
  $$\mu(x) = \max(1 - |1 - x|, 0),$$
  and
- the quantity $y$ is described by the membership function
  $$\mu(y) = \max(1 - |1 - y|, 0).$$

Use the values $\alpha = 0.25, 0.5, 0.75,$ and $1.0$ to describe membership functions for $z = x - y$ and $t = x \cdot y$.

**Computing $\alpha$-cuts for $x$.**

- For $\alpha = 0.25$, the condition $\max(1 - |1 - x|, 0) \geq 0.25$ means that $1 - |1 - x| \geq 0.25$, so $|1 - x| = |x - 1| \leq 1 - 0.25 = 0.75$, i.e., $-0.75 \leq x - 1 \leq 0.75$. Adding 1 to all three sides of the inequality, we get $0.25 \leq x \leq 1.75$. Thus, the corresponding $\alpha$-cut is $[0.25, 1.75]$.

- For $\alpha = 0.5$, the condition $1 - |1 - x| \geq 0.5$ means $|1 - x| = |x - 1| \leq 1 - 0.5 = 0.5$, i.e., $-0.5 \leq x - 1 \leq 0.5$. Adding 1 to all three sides of the inequality, we get $0.5 \leq x \leq 1.5$. Thus, the corresponding $\alpha$-cut is $[0.5, 1.5]$.

- For $\alpha = 0.75$, the condition $1 - |1 - x| \geq 0.75$ means $|1 - x| = |x - 1| \leq 1 - 0.75 = 0.25$, i.e., $-0.25 \leq x - 1 \leq 0.25$. Adding 1 to all three sides of the inequality, we get $0.75 \leq x \leq 1.25$. Thus, the corresponding $\alpha$-cut is $[0.75, 1.25]$.

- For $\alpha = 1$, the condition $1 - |1 - x| \geq 1$ means $|1 - x| = |x - 1| \leq 1 - 1 = 0$, i.e., $0 \leq x - 1 \leq 0$. Adding 1 to all three sides of the inequality, we get $1 \leq x \leq 1$. Thus, the corresponding $\alpha$-cut is $[1, 1]$.

**Computing $\alpha$-cuts for $y$.**
For $\alpha = 0.25$, the condition $1 - |1 - y| \geq 0.25$ means

$$|1 - y| = |y + 1| \leq 1 - 0.25 = 0.75,$$

i.e., $-0.75 \leq y + 1 \leq 0.75$. Subtracting 1 from all three sides of the inequality, we get $-1.75 \leq y \leq -0.25$. Thus, the corresponding $\alpha$-cut is

$$[-1.75, -0.25].$$

For $\alpha = 0.5$, the condition $1 - |1 - y| \geq 0.5$ means

$$|1 - y| = |y + 1| \leq 1 - 0.5 = 0.5,$$

i.e., $-0.5 \leq y + 1 \leq 0.5$. Subtracting 1 from all three sides of the inequality, we get $-1.5 \leq y \leq -0.5$. Thus, the corresponding $\alpha$-cut is

$$[-1.5, -0.5].$$

For $\alpha = 0.75$, the condition $1 - |1 - y| \geq 0.75$ means

$$|1 - y| = |y + 1| \leq 1 - 0.75 = 0.25,$$

i.e., $-0.25 \leq y + 1 \leq 0.25$. Subtracting 1 from all three sides of the inequality, we get $-1.25 \leq y \leq -0.75$. Thus, the corresponding $\alpha$-cut is

$$[-1.25, -0.75].$$

For $\alpha = 1$, the condition $1 - |1 - y| \geq 1$ means

$$|1 - y| = |y + 1| \leq 1 - 1 = 0,$$

i.e., $0 \leq y + 1 \leq 0$. Subtracting 1 from all three sides of the inequality, we get $-1 \leq y \leq -1$. Thus, the corresponding $\alpha$-cut is

$$[-1, -1].$$

Computing $\alpha$-cuts for $z = x - y$.

- For $\alpha = 0.25$, we have

$$[0.25, 1.75] - [-1.75, -0.25] = [0.25 - (-0.25), 1.75 - (-1.75)] = [0.5, 3.5].$$

- For $\alpha = 0.5$, we have

$$[0.5, 1.5] - [-1.5, -0.5] = [0.5 - (-0.5), 1.5 - (-1.5)] = [1, 3].$$

- For $\alpha = 0.75$, we have

$$[0.75, 1.25] - [-1.25, -0.75] = [0.75 - (-0.75), 1.25 - (-1.25)] = [1.5, 2.5].$$
• For $\alpha = 1$, we have

$$[1, 1] - [-1, -1] = [1 - (-1), 1 - (-1)] = [2, 2].$$

**Computing $\alpha$-cuts for $t = x \cdot y$.**

• For $\alpha = 0.25$, we have

$$[0.25, 1.75] \cdot [-1.75, -0.25] =

[\min(0.25 \cdot (-1.75), 0.25 \cdot (-0.25), 1.75 \cdot (-1.75), 1.75 \cdot (-0.25)),
\max(0.25 \cdot (-1.75), 0.25 \cdot (-0.25), 1.75 \cdot (-1.75), 1.75 \cdot (-0.25))]

= [\min(-0.4375, -0.0625, -3.0625, -0.4375), \max(-0.4375, -0.0625, -3.0625, -0.4375)]
= [-3.0625, -0.0625].$$

• For $\alpha = 0.5$, we have

$$[0.5, 1.5] \cdot [-1.5, -0.5] =

[\min(0.5 \cdot (-1.5), 0.5 \cdot (-0.5), 1.5 \cdot (-1.5), 1.5 \cdot (-0.5)),
\max(0.5 \cdot (-1.5), 0.5 \cdot (-0.5), 1.5 \cdot (-1.5), 1.5 \cdot (-0.5))]

= [\min(-0.75, -0.25, -2.25, -0.75), \max(-0.75, -0.25, -2.25, -0.75)]
= [-2.25, -0.25].$$

• For $\alpha = 0.75$, we have

$$[0.75, 1.25] \cdot [-1.25, -0.75] =

[\min(0.75 \cdot (-1.25), 0.75 \cdot (-0.75), 1.25 \cdot (-1.25), 1.25 \cdot (-0.75)),
\max(0.75 \cdot (-1.25), 0.75 \cdot (-0.75), 1.25 \cdot (-1.25), 1.25 \cdot (-0.75))]

= [\min(-0.9375, -0.5625, -1.5625, -0.9375), \max(-0.9375, -0.5625, -1.5625, -0.9375)]
= [-1.5625, -0.9375].$$

• For $\alpha = 1$, we have

$$[1, 1] \cdot [-1, -1] = [-1, -1].$$