Solution to Problem 8

Task. Check and, if possible, use monotonicity to estimate the ranges of the functions from Problems 1 and 2.

Problem 1. Here, $f'(x) = -2 - 2x$. Thus, the interval estimate of the range of this derivative when $x \in [0, 6]$ is

$$f'([0,6]) = -2 - 2 \cdot [0,6] = [-2,-2] - [2,2] \cdot [0,6] = [-2,-2] - [0,12] =$$

$$[-2 - 12, -2 - 0] = [-14, -2].$$

The upper endpoint is negative, so this range includes only negative values of the derivative. This means that the function is decreases. Thus, its range is equal to $f([0,6]) = [f(6), f(0)]$. Here,

$$f(6) = (4 + 6) \cdot (2 - 6) = 10 \cdot (-4) = -40$$

and

$$f(0) = (4 + 0) \cdot (2 - 0) = 4 \cdot 2 = 8.$$

Thus, the range of the function $f(x)$ on the given interval $[0, 6]$ is equal to $[-40, 8]$.

Problem 2. Here,

$$\frac{\partial f}{\partial x_1} = -4 - x_2,$$

so the range of this derivative is

$$-4 - [-1, 6] = [-4, -4] - [-1, 6] = [-4 - 6, -4 - (-1)] = [-10, -3].$$

This range only contains negative numbers, so the function is decreasing in terms of x_1. Thus:

- To find the smallest value of the function $f(x_1, x_2)$, it is sufficient to consider the largest possible value of x_1, i.e., $x_1 = 6$. For this value x_1, the function takes the form

$$f(0, x_2) = (2 - 6) \cdot (4 + x_2) = -4 \cdot (4 + x_2).$$

- To find the largest value of the function $f(x_1, x_2)$, it is sufficient to consider the smallest possible value of x_1, i.e., $x_1 = 0$. For this value x_1, the function takes the form

$$f(0, x_2) = 2 \cdot (4 + x_2).$$

Here:

- For the function $f(6, x_2) = -4 \cdot (4 + x_2)$, the derivative with respect to x_2 is equal to -4 and is thus, negative. Thus, this function is decreasing with respect to x_1, and its smallest possible value is attained when x_2 is the largest, i.e., when $x_2 = 6$. For this value x_2, we get

$$\underline{y} = f(6, 6) = -4 \cdot (4 + 6) = -4 \cdot 10 = -40.$$

- For the function $f(0, x_2) = 2 \cdot (4 + x_2)$, the derivative with respect to x_2 is equal to 2 and is, thus, positive. Thus, this function is increasing with respect to x_1, and its largest possible value is attained when x_2 is the largest, i.e., when $x_2 = 6$. For this value x_2, we get

$$\overline{y} = f(0, 6) = 2 \cdot (4 + 6) = 2 \cdot 10 = 20.$$

So, the range $[\underline{y}, \overline{y}]$ of the function $f(x_1, x_2)$ on the given intervals is equal to

$$[\underline{y}, \overline{y}] = [-40, 20].$$