Solution to Problem 9

Task. Use monotonicity and one iteration of bisection to estimate the range of the functions from Problems 1 when x is in the interval $[-4, 0]$.

Solution. For this problem, bisection means that instead of the original interval $[-4, 0]$, we consider two subintervals $[-4, -2]$ and $[-2, 0]$, and take the union of the resulting two ranges.

Interval $[-4, -2]$. For this interval, the range of the derivative is equal to

$$-2 - 2 \cdot [-4, -2] = [-2, -2] - [-8, -4] = [2, 10].$$

All the values in this range are positive, so the function is increasing. Thus:

- its largest value is attained when x is the largest $x = -2$, the value $f(-2)$ is
$$f(-2) = (4 + (-2)) \cdot (2 - (-2)) = 2 \cdot 4 = 8;$$
and
- its smallest value is attained when x is the smallest $x = -4$, the value $f(-4)$ is
$$f(-4) = (4 + (-4)) \cdot (2 - (-4)) = 0 \cdot 6 = 0.$$

Thus, for this interval, we get the range $[0, 8]$.

Interval $[-2, -0]$. For this interval, the range of the derivative $f'(x) = -2 - 2x$ is equal to

$$-2 - 2 \cdot [-2, 0] = [-2, -2] - [-4, 0] = [-2, 2].$$

This range includes both positive and negative values, so we cannot conclude that the function is monotonic. Straightforward interval computations lead to

$$(4 + [-2, 0]) \cdot (2 - [-2, 0]) = [2, 4] \cdot [2, 4] = [4, 16].$$

The resulting range. The resulting range is estimated as the union

$$[0, 8] \cup [4, 16]$$

of the ranges estimated for subintervals. This union is equal to

$$[\min(4, 0), \max(8, 16)] = [0, 16].$$