

Solutions to Test 2, Interval Computations

Fall 2023

Problem 1. Why do we need guaranteed bounds?

Solution. In many practical situations, it is critically important to guarantee bounds:

- In chemical engineering, it is important to guarantee that the level of the pollutants does not exceed the legal thresholds, otherwise the plant will face a huge fine and may even be closed.
- In planning a trajectory of a space flight to the Moon, it is important that the spacecraft ends up on the Moon surface and not nearby.
- In nuclear engineering, it is absolutely critical to make sure that the system does not go beyond the critical level – otherwise, we have a disaster on our hands.

Problem 2. Who invented interval computations?

Solution. Main ideas can be traced to Archimedes, but seriously this field was started, in the late 1950s, by three independent researchers:

- Ramon Moore in the US,
- Mieczyslaw Warmus in Poland, and
- Teruo Sunaga in Japan.

3–6. Let us use different methods to estimate the range of the function $f(x) = x^2 + x + 1$ when $x \in [-0.6, -0.2]$.

Problem 3. Use straightforward interval computations.

Solution. By applying straightforward interval computations, we get

$$\begin{aligned} [-0.6, -0.2] \cdot [-0.6, -0.2] + [-0.6, -0.2] + 1 &= [0.04, 0.36] + [-0.6, -0.2] + 1 = \\ &= [0.04 - 0.6 + 1, 0.36 - 0.2 + 1] = [0.44, 1.16]. \end{aligned}$$

Problem 4. Check for monotonicity.

Solution. The derivative is equal to $f'(x) = 2x + 1$. The range of the derivative when $x \in [-0.6, -0.2]$ is equal to:

$$f'([-0.6, -0.2]) = 2 \cdot [-0.6, -0.2] + 1 = [-1.2, -0.4] + 1 = [-0.2, 0.6].$$

This range includes both positive and negative values, so we cannot conclude that the function is monotonic on the given interval.

Problem 5. Apply bisection and try to use monotonicity for each of the two resulting problems.

Solution. Bisection means dividing the original interval $[-0.6, -0.2]$ into two equal intervals $[-0.6, -0.4]$ and $[-0.4, -0.2]$.

On the second of these intervals,

$$f'([-0.4, -0.2]) = 2 \cdot [-0.4, -0.2] + 1 = [-0.8, -0.4] + 1 = [0.2, 0.6].$$

All these values are positive, so the function $f(x)$ is increasing on this interval. Thus, on this interval:

- the function attains its largest value when x is the largest, i.e., when $x = -0.2$; in this case,

$$f(-0.2) = (-0.2)^2 + (-0.2) + 1 = 0.04 + 0.2 + 1 = 0.84;$$

- the function attains its smallest value when x is the smallest, i.e., when $x = -0.4$; in this case,

$$f(-0.4) = (-0.4)^2 + (-0.4) + 1 = 0.16 - 0.4 + 1 = 0.76.$$

So, the range of the function $f(x)$ on the interval $[-0.4, -0.2]$ is equal to $[0.76, 0.84]$.

On the first of the new intervals, we get

$$f'([-0.6, -0.4]) = 2 \cdot [-0.6, -0.4] + 1 = [-1.2, -0.8] + 1 = [-0.2, 0.2].$$

This range includes both positive and negative values, so we cannot conclude that the function is monotonic on this interval.

Problem 6. Use the centered form for those intervals on which we do not have monotonicity.

Solution. For the interval $[-0.6, -0.4]$, the midpoint is $\tilde{x}_1 = \frac{-0.6 + (-0.4)}{2} = -0.5$, and $\Delta_1 = \frac{-0.4 - (-0.6)}{2} = 0.1$. We have

$$\tilde{y} = f(\tilde{x}_1) = (-0.5)^2 + (-0.5) + 1 = 0.25 - 0.5 + 1 = 0.75,$$

and we know that $f'([-0.6, -0.4]) = [-0.2, 0.2]$. So, the centered form leads to the following estimate for the range:

$$\begin{aligned}\tilde{y} + f'([-0.6, -0.4]) \cdot [-\Delta_1, \Delta_1] &= 0.75 + [-0.2, 0.2] \cdot [-0.1, 0.1] = \\ &= 0.75 + [-0.02, 0.02] = [0.73, 0.77].\end{aligned}$$

To get the range of $f(x)$ over the whole original interval $[0.2, 0.6]$, we need to take the union of ranges over two subintervals:

$$[0.76, 0.84] \cup [0.73, 0.77] = [0.73, 0.84].$$

7–10. Let us use different methods to estimate the range of the function $f(x_1, x_2) = x_1 \cdot x_2 + 0.5 \cdot x_1 + 0.5 \cdot x_2$ when $x_1 \in [-0.6, -0.2]$ and $x_2 \in [-0.8, -0.4]$.

Problem 7. Use straightforward interval computations.

Solution.

$$\begin{aligned} & [-0.6, -0.2] \cdot [-0.8, -0.4] + 0.5 \cdot [-0.6, -0.2] + 0.5 \cdot [-0.8, -0.4] = \\ & [0.08, 0.48] + [-0.3, -0.1] + [-0.4, -0.2] = [0.08 - 0.3 - 0.4, 0.48 - 0.1 - 0.2] = \\ & \quad [-0.62, 0.18]. \end{aligned}$$

Problem 8. Check for monotonicity with respect to each of the variables.

Solution. Here,

$$\frac{\partial f}{\partial x_1} = x_2 + 0.5 \text{ and } \frac{\partial f}{\partial x_2} = x_1 + 0.5.$$

So,

$$\frac{\partial f}{\partial x_1}([-0.6, -0.2], [-0.8, -0.4]) = [-0.8, -0.4] + 0.5 = [-0.3, 0.1]$$

and

$$\frac{\partial f}{\partial x_2}([-0.6, -0.2], [-0.8, -0.4]) = [-0.6, -0.2] + 0.5 = [-0.1, 0.3].$$

Each of these intervals includes both positive and negative values, so we cannot conclude that the function is monotonic with respect to any of its variables.

Problem 9. Apply bisection. Use the algorithm we studied in class to decide which side to bisect. If both sides are equally appropriate, bisect by x_1 .

Solution. Here,

$$\tilde{x}_1 = \frac{-0.6 + (-0.2)}{2} = -0.4, \quad \Delta_1 = \frac{-0.2 - (-0.6)}{2} = 0.2,$$

$$\tilde{x}_2 = \frac{-0.8 + (-0.4)}{2} = -0.6, \quad \text{and } \Delta_2 = \frac{-0.4 - (-0.8)}{2} = 0.2.$$

At the point $(\tilde{x}_1, \tilde{x}_2) = (-0.4, -0.6)$, the derivatives $c_i = \frac{\partial f}{\partial x_i}$ are equal to

$$c_1 = \tilde{x}_2 + 0.5 = -0.6 + 0.5 = -0.1 \text{ and } c_2 = \tilde{x}_1 + 0.5 = -0.4 + 0.5 = -0.1.$$

According to the algorithm, we must select the side i with the largest value of the product $|c_i| \cdot \Delta_i$. Here,

$$|c_1| \cdot \Delta_1 = |-0.1| \cdot 0.2 = 0.02 \text{ and } |c_2| \cdot \Delta_2 = |0.1| \cdot 0.2 = 0.02.$$

These products are equal, so we bisect the interval for x_1 , into intervals $[0.2, 0.4]$ and $[0.4, 0.6]$.

Problem 10. Use monotonicity to estimate the range of each of the two resulting problems; use centered form when there is no monotonicity, and then use the resulting estimates to find the range over all possible pairs (x_1, x_2) .

Solution. On the second interval $[-0.4, -0.2]$, we have

$$\frac{\partial f}{\partial x_2}([-0.4, -0.2], [-0.8, -0.4]) = [-0.4, -0.2] + 0.5 = [0.1, 0.3].$$

All the values from this interval are positive, so the function f is increasing with respect to x_2 . So:

- to find the largest value $\bar{y}$ on this interval, it is sufficient to consider the largest value $x_2 = -0.4$, and
- to find the smallest value $\underline{y}$ on this interval, it is sufficient to consider the smallest value $x_2 = -0.8$.

Here:

- When $x_2 = -0.4$, the function takes the form

$$x_1 \cdot (-0.4) + 0.5 \cdot x_1 + 0.5 \cdot (-0.4) = 0.1 \cdot x_1 - 0.2,$$

so its range is equal to

$$0.1 \cdot [-0.4, -0.2] - 0.2 = [-0.04, -0.02] - 0.2 = [-0.24, -0.22].$$

The largest value is -0.22 .

- When $x_2 = -0.8$, the function takes the form

$$x_1 \cdot (-0.8) + 0.5 \cdot x_1 + 0.5 \cdot (-0.8) = -0.3 \cdot x_1 - 0.4,$$

so its range is equal to

$$-0.3 \cdot [-0.4, -0.2] - 0.4 = [0.06, 0.12] - 0.4 = [-0.34, -0.28].$$

The smallest value is -0.34 .

Thus, the range of f on this interval is $[-0.34, -0.22]$.

For the interval $[-0.6, -0.4]$, we have

$$\frac{\partial f}{\partial x_2}([-0.6, -0.4], [-0.8, -0.4]) = [-0.6, -0.4] + 0.5 = [-0.1, 0.1].$$

This interval includes both positive and negative values, so we cannot conclude that the function is monotonic. Let us use centered form. Here,

$$\tilde{x}_1 = \frac{-0.6 + (-0.4)}{2} = -0.5, \quad \Delta_1 = \frac{-0.4 - (-0.6)}{2} = 0.1,$$

$$\tilde{x}_2 = \frac{-0.8 + (-0.4)}{2} = -0.6, \quad \Delta_2 = \frac{-0.4 - (-0.8)}{2} = 0.2.$$

Here,

$$\begin{aligned} \tilde{y} = f(\tilde{x}_1, \tilde{x}_2) &= f(-0.5, -0.6) = (-0.5) \cdot (-0.6) + 0.5 \cdot (-0.5) + 0.5 \cdot (-0.6) = \\ &= 0.3 - 0.25 - 0.3 = -0.25. \end{aligned}$$

Here, we already know that

$$\frac{\partial f}{\partial x_1}([-0.6, -0.4], [-0.8, -0.4]) = [-0.1, 0.3]$$

and

$$\frac{\partial f}{\partial x_2}([-0.6, -0.4], [-0.8, -0.4]) = [-0.1, 0.1].$$

Thus, the centered form leads to the following estimate for $f([0.4, 0.6], [0.4, 0.8])$:

$$\begin{aligned} -0.25 + [-0.1, 0.3] \cdot [-0.1, 0.1] &= [-0.1, 0.1] \cdot [-0.2, 0.2] = \\ &= -0.25 + [-0.03, 0.03] + [-0.02, 0.02] = [-0.3, -0.2]. \end{aligned}$$

Taking the union of this range and the range $[-0.34, -0.22]$, we get

$$[-0.34, -0.22] \cup [-0.3, -0.2] = [-0.34, -0.2].$$