

Solutions to Test 3 for Interval Class, Fall 2023

Problem 1. Use the interval-based optimization algorithm to locate the maximum of the function $x^2 + x + 1$ on the interval $[-0.6, -0.2]$. Divide this interval into two, then divide the remaining intervals into two again, etc. At each iteration, dismiss subintervals where maximum cannot be attained. Stop when you get intervals of width 0.1.

Solution. Let us first divide this interval into two subintervals: $[-0.6, -0.4]$ and $[-0.4, -0.2]$.

On the second interval, the range of the derivative $2x + 1$ is equal to

$$2 \cdot [-0.4, -0.2] + 1 = [-0.8, -0.4] + 1 = [0.2, 0.6].$$

All these values are positive, so the function is increasing on this subinterval.

- Its smallest value is attained when x_1 is the smallest $x_1 = -0.4$ and is equal to

$$f(-0.4) = (-0.4)^2 + (-0.4) + 1 = 0.16 - 0.4 + 1 = -0.24 + 1 = 0.76.$$

- Its largest value is attained when x_1 is the largest $x_1 = -0.2$ and is equal to

$$f(-0.2) = (-0.2)^2 + (-0.2) + 1 = 0.04 - 0.2 + 1 = -0.16 + 1 = 0.84.$$

Thus, the range of the function on the first subinterval is $[0.76, 0.84]$.

For the first subinterval, the range of the derivative is equal to

$$2 \cdot [-0.6, -0.4] - 1 = [-1.2, -0.8] + 1 = [-0.2, 0.2].$$

This range contains both positive and negative values, so we have to use the centered form. For this subinterval, the midpoint is $\tilde{x} = -0.5$. At this point, the value of the function is

$$f(-0.5) = (-0.5)^2 + (-0.5) + 1 = 0.25 - 0.5 + 1 = -0.25 + 1 = 0.75.$$

The half-widths of this subinterval is $\Delta = 0.1$. Thus, the centered form leads to the following enclosure for the range:

$$f(\tilde{x}) + f'([\underline{x}, \bar{x}]) \cdot [-\Delta, \Delta] = 0.75 + [-0.2, 0.2] \cdot [-0.1, 0.1] =$$

$$0.75 + [-0.02, 0.02] = [0.73, 0.77].$$

The largest value from this range is smaller than one of the values $f(0.2) = 0.84$, which means that the maximum cannot be attained on the first subinterval.

Thus, the maximum is attained on the second subinterval. Since on the second subinterval, the function is increasing, its maximum there is attained at the largest value $x = -0.2$. Thus, the maximum of the function $f(x)$ on the whole interval $[-0.6, -0.2]$ is attained when $x = -0.2$.

Problem 2–3. Use the constraints method to solve the following two problems:

- find x_1 from the interval $[0, 1]$ and x_2 from the interval $[0, 1]$, for which $x_1 + x_2 = 1$ and $x_1 \cdot x_2 = 0.25$;
- find x_1 and x_2 from the same intervals, for which $x_1 + x_2 = 1$ and $x_1 \cdot x_2 = 0.5$.

For each problem, perform two cycles, in each of which you apply all the rules one by one.

First system of equations. Based on the equations, we form the following four rules:

1. $x_1 \mapsto x_2 = 1 - x_1$;
2. $x_2 \mapsto x_1 = 1 - x_2$;
3. $x_1 \mapsto x_2 = 0.25/x_1$;
4. $x_2 \mapsto x_1 = 0.25/x_2$.

We start with $x_1 \in \mathbf{x}_1 = [0, 1]$ and $x_2 \in \mathbf{x}_2 = [0, 1]$. We apply Rules 1, 2, 3, 4, then again 1, 2, 3, 4, etc. If we get stuck, we bisect one of the intervals.

- Rules 1 and 2 do not change anything.
- Rule 3 leads to $\mathbf{x}_2 = [0.25, \infty) \cap [0, 1] = [0.25, 1]$.
- Rule 4 leads to $\mathbf{x}_1 = [0.25, 1] \cap [0, 1] = [0.25, 1]$.
- Rule 1 leads to $\mathbf{x}_2 = (1 - [0.25, 1]) \cap [0.25, 1] = [0, 0.75] \cap [0.25, 1] = [0.25, 0.75]$.
- Rule 2 leads to $\mathbf{x}_1 = (1 - [0.25, 0.75]) \cap [0.25, 1] = [0.25, 0.75] \cap [0.25, 1] = [0.25, 0.75]$.
- Rule 3 leads to $\mathbf{x}_2 = (0.25 \cdot [1.33 \dots, 4]) \cap [0.25, 0.75] = [0.33 \dots, 1] \cap [0.25, 0.75] = [0.33 \dots, 0.75]$.
- Rule 4 leads to $\mathbf{x}_1 = (0.25 \cdot [1.33 \dots, 3]) \cap [0.25, 0.75] = [0.33 \dots, 0.75] \cap [0.25, 0.75] = [0.33 \dots, 0.75]$.

Second system of equations. Based on the equations, we form the following four rules:

1. $x_1 \mapsto x_2 = 1 - x_1$;
2. $x_2 \mapsto x_1 = 1 - x_2$;
3. $x_1 \mapsto x_2 = 0.5/x_1$;
4. $x_2 \mapsto x_1 = 0.5/x_2$.

We start with $x_1 \in \mathbf{x}_1 = [0, 1]$ and $x_2 \in \mathbf{x}_2 = [0, 1]$. We apply Rules 1, 2, 3, 4, etc.

- Rules 1 and 2 do not change anything.
- Rule 3 leads to $\mathbf{x}_2 = [0.5, \infty) \cap [0, 1] = [0.5, 1]$.
- Rule 4 leads to $\mathbf{x}_1 = (0.5 \cdot [1, 2]) \cap [0, 1] = [0.5, 1] \cap [0, 1] = [0.5, 1]$.
- Rule 1 leads to $\mathbf{x}_2 = (1 - [0.5, 1]) \cap [0.5, 1] = [0, 0.5] \cap [0.5, 1] = [0.5, 0.5]$.
- Rule 2 leads to $\mathbf{x}_1 = (1 - [0.5, 0.5]) \cap [0.5, 1] = [0.5, 0.5] \cap [0.5, 1] = [0.5, 0.5]$.
- Rule 3 leads to $\mathbf{x}_2 = (0.5 \cdot [2, 2]) \cap [0.5, 0.5] = [1, 1] \cap [0.5, 0.5] = \emptyset$.

So, this system does not have any solutions.

Problem 4. For the function $x_1 \cdot x_2 + 0.5 \cdot x_1 + 0.5 \cdot x_2$, when $\tilde{x}_1 = -0.4$ and $\tilde{x}_2 = -0.6$, and the standard deviations are $\sigma_1 = \sigma_2 = 0.1$, estimate the variance σ^2 of the result of data processing.

Solution. The general formula is

$$c^2 = \sum_{i=1}^n c_i^2 \cdot \sigma_i^2, \text{ where } c_i = \frac{\partial f}{\partial x_i}(\tilde{x}_1, \dots, \tilde{x}_n).$$

In our case,

$$\frac{\partial f}{\partial x_1} = x_2 + 0.5, \text{ so } c_1 = -0.4 + 0.5 = 0.1, \text{ and}$$

$$\frac{\partial f}{\partial x_2} = x_1 + 0.5, \text{ so } c_2 = -0.6 + 0.5 = -0.1.$$

Thus,

$$\begin{aligned} c^2 &= 0.1^2 \cdot 0.1^2 + (-0.1)^2 \cdot 0.1^2 = 0.01 \cdot 0.01 + 0.01 \cdot 0.01 = \\ &0.0001 + 0.0001 = 0.0002. \end{aligned}$$

Problem 5. Write down:

- an expression for which an optimizing compiler improves the estimates provided by straightforward interval computations, and
- an expression for which an optimizing compiler worsens improves the estimates provided by straightforward interval computations.

Your examples should be different from examples given in class (similar is OK).

First example. Let us consider the problem of estimating the expression $a \cdot b + a \cdot c$ when $a \in [-2, 2]$, $b = 2$ (i.e., $b \in [2, 2]$) and $c \in [-2, -1]$.

In this example, straightforward interval computations lead to

$$[-2, 2] \cdot [2, 2] + [-2, 2] \cdot [-2, -1] = [-4, 4] + [-4, -2] = [-8, 2].$$

For this example, optimizing compiler – aiming to minimize the number of multiplications – will transform the original expression into $a \cdot (b + c)$. For this new expression, straightforward interval computations lead to a narrower interval:

$$[-2, 2] \cdot ([2, 2] + [-2, -1]) = [-2, 2] \cdot [0, 1] = [-2, 2].$$

Second example. Let us consider the problem of estimating the expression

$$\frac{1}{1 + \frac{a}{b}}$$

when $\mathbf{a} = \mathbf{b} = [1, 2]$. In this case, straightforward interval computations lead to

$$\frac{1}{1 + \frac{[1, 2]}{[1, 2]}} = \frac{1}{1 + [0.5, 2]} = \frac{1}{[1.5, 3]} = [0.33 \dots, 0.66 \dots].$$

For this example, optimizing compiler – aiming to minimize number of divisions – will transform the original expression into

$$\frac{b}{a + b}.$$

For this new expression, straightforward interval computations lead to a wider interval:

$$\frac{[1, 2]}{[1, 2] + [1, 2]} = \frac{[1, 2]}{[2, 4]} = [0.25, 1].$$

Problem 6. If we only use numbers with one digit after the decimal point, and we use rounding that preserves guaranteed bounds, what will be the result of multiplying the intervals $[0.5, 0.7]$ and $[0.3, 0.6]$?

Solution. The exact product of the two intervals is

$$[0.5 \cdot 0.7, 0.6 \cdot 0.6] = [0.15, 0.42].$$

To get guaranteed bounds, we need to round down the lower endpoint, to 0.1, and round up the upper bound, to 0.5. Thus, the resulting interval is $[0.1, 0.5]$.

Problem 7. Two questions:

- If out of 20 experts, 16 think that the statement is true, what degree of confidence shall we assign to this statement?
- If an expert marked her confidence in a statement as 4.5 on a scale from 0 to 5, what degree of confidence shall we assign to this statement?

Solution. In general, if m out of n experts believe that the given statement is true, we assign, to this statement, the degree of confidence m/n . In our case, we assign the degree $16/20 = 0.8$.

In general, if an expert marks his/her confidence in a statement as m on a scale from 0 to n , we assign, to this statement, the degree of confidence m/n . In our case, we assign the degree $4.5/5 = 0.9$.

Problem 8. If we have $\mu(0) = 1$ and $\mu(5) = 0$, what value shall we assign to $\mu(2)$? Use linear interpolation.

Solution. If we know the value $y_1 = f(x_1)$ and the value $y_2 = f(x_2)$, then linear interpolation means estimating $f(x)$ as

$$f(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1).$$

In our case, $x_1 = 0$, $y_1 = 1$, $x_2 = 5$, $y_2 = 0$, and $x = 2$. So, we get

$$\mu(2) = 1 + \frac{0 - 1}{5 - 0} \cdot (2 - 0) = 1 - 0.2 \cdot 2 = 1 - 0.4 = 0.6.$$

Problem 9. If the expert's degrees of confidence in two statements A and B are 0.7 and 0.9, and we use $\min$ as an “and”-operation and $\max$ as an “or”-operation, what degree of confidence shall we assign to $A \& B$? To $A \vee B$? If a statement C has degree of confidence 0.8, what is our degree of belief in $(A \& B) \vee C$?

Solution. For $A \& B$, we get $d(A \& B) = \min(0.7, 0.9) = 0.7$. Similarly,

$$d(A \vee B) = \max(0.7, 0.9) = 0.9,$$

and

$$d((A \& B) \vee C) = \max(d(A \& B), d(C)) = \max(0.7, 0.8) = 0.8.$$

Problem 10. Suppose that we have three alternatives, for which the gains are in the intervals $[0, 50]$, $[20, 30]$, and $[30, 40]$.

- is there an alternative which is guaranteed to be optimal (i.e., for which the gain is the largest)?
- list all alternatives which can be optimal;
- which alternative(s) should we select if we use Hurwicz optimism-pessimism criterion with $\alpha = 0, 0.5$, and 1 .

Solution.

- An alternative $[\underline{x}_i, \bar{x}_i]$ is guaranteed to be optimal if $\underline{x}_i \geq \bar{x}_j$ for all $j \neq i$. This is not true in this case:

$$\underline{x}_1 = 0 \not\geq \bar{x}_2 = 30, \underline{x}_2 = 20 \not\geq \bar{x}_1 = 50, \text{ and } \underline{x}_3 = 30 \not\geq \bar{x}_1 = 50.$$

- An alternative $[\underline{x}_i, \bar{x}_i]$ is possibly optimal if $\bar{x}_i \geq \max_j \underline{x}_j$. In our case, $\max_j \underline{x}_j = \max(0, 20, 30) = 30$. So, this inequality is satisfied for $i = 1$ and $i = 3$. So, the first and third alternatives are possibly optimal.
- In general, we compare the values $x_i = \alpha \cdot \bar{x}_i + (1 - \alpha) \cdot \underline{x}_i$. Here are the values for different α and i . For each α , the alternative with the largest value of x_i is underlined:

	$i = 1$	$i = 2$	$i = 3$
$\alpha = 0$	0	20	<u>30</u>
$\alpha = 0.5$	25	25	<u>35</u>
$\alpha = 1$	<u>50</u>	30	40