9/2/8

<u>Reminder of simple prob. things</u>

- How to get estimates
- How to process uncertainty
- How to deal with Interval Uncertainty.

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right)$$

Prob. Density For Gauss Distr.

$$\text{Prob}\ (x \le \xi \le x + \Delta x) \approx \rho(x) \cdot \Delta x$$

random variable

$$\rho(x) = \frac{\text{Prob}\ (x \le \xi \le x + \Delta x)}{\Delta x}$$

Given: some measurments $x_1, \dots, x_n$

<u>Solar:</u> We Know a, σ We simulate the distribution and predict the probability at dif points.

? How do we find a and σ?

<u>Idea</u> is to select the most probable values a, σ.

e.g.

$H_1 = a = 0, \quad \sigma = 1 \qquad x_1 \approx 0$

$H_2 = a = 20, \quad \sigma = 1$

If we use hypotesis 1 (H_1)

$$\cancel{\frac{1}{\sqrt{2\pi} \cdot 1} \cdot \exp(-0^2)} = 0$$

If we use hypo 2 (H_2)

$$\frac{1}{\sqrt{2\pi} \cdot 1} \cdot \exp(-200)$$

200

Depends accuracy of the measurement

1st Observation

$$\frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{(x_1-a)^2}{2\sigma^2}\right) \Delta x$$

2nd observation

$$\frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{(x_2-a)^2}{2\sigma^2}\right) \Delta x$$

$$\vdots$$

nth observation

$$\frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{(x_n-a)^2}{2\sigma^2}\right) \Delta x$$

⚠ We multiply

$$\left(\frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(\frac{-(x_1-a)^2}{2\sigma^2}\right) \cdot \Delta x\right) \cdot \left(\frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(\frac{-(x_2-a)^2}{2\sigma^2}\right) \cdot \Delta x\right) \cdots \left(\frac{1}{2\pi\sigma} \cdot \exp\left(\frac{-(x_n-a)^2}{2\sigma^2}\right) \Delta x\right)$$

$$\max_{a,\sigma}$$

⚠ Tricks!

$$a^b \cdot a^c = a^{(b+c)}$$

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma^n} \cdot \exp\left(\frac{-(x_1-a)^2}{2\sigma^2}\right) \cdot \exp\left(\frac{-(x_2-a)}{2\sigma^2}\right) \cdots \exp\left(\frac{-(x_n-a)}{2\sigma^2}\right)$$

$$\text{Const}\ \frac{1}{\sigma^n}\ \exp\left(\frac{-(x_1-a)^2}{2\sigma^2} - \frac{(x_1-a)^2}{2\sigma^2} - \cdots - \frac{(x_n-a)^2}{2\sigma^2}\right) \to \max_{a,\sigma}$$

Ex, Deruvide with respect to a

$$-\frac{(x_1-a)^2}{2\sigma^2} - \frac{(X_2-a)^2}{2\sigma^2} - \quad - \frac{(x_n-a)^2}{2\sigma^2} \rightarrow \max_{a,\sigma}$$

Constant

$\frac{\partial x_i}{\partial x a} = 0$

$\frac{\partial a}{\partial a} = 1$

! We can take the constants out $\left(\frac{1}{2\sigma^2}\right)$, then deruvete the inner terms.

$$\frac{1}{2\sigma^2}\left(-2(x_1-a)(-1) + 2(X_2-a)(-1) + \ldots + 2(X_n-a)(-1)\right) = 0$$

⚠ If we multiply both sides by same constant remains true so we got:

$$(x-a)+(x_2-a)+ \ldots + (x_n-a) = 0$$

How we find σ?

$$-\frac{(x_1-a)^2}{2\sigma^2} - \ldots - \frac{(x_n-a)^2}{2\sigma^2} - n\ln\sigma \rightarrow \max\ a,\sigma$$

$$\frac{(x_1-a)^2}{2\sigma^2} + \ldots + \frac{(x_n-a)^2}{2\sigma^2} + n\ln\sigma \rightarrow \min\ a,\sigma$$

Now let's diff with rosped σ

$$\frac{(x_1-a)^2}{2\sigma^2} + \dots + \frac{(x_n-a)^2}{2\sigma^2} + n\ln\sigma \to \max_{a,\sigma}$$

$$(x_1-a)^2\frac{\sigma^{-2}}{2} + \dots + (x_n-a)^2\frac{\sigma^{-2}}{2} + n\ln\sigma \to \min_{a,\sigma}$$

$$\frac{-\cancel{2}\sigma^{-3}}{\cancel{2}}(x_1-a)^2 + \dots + \frac{-\cancel{2}\sigma^{-3}}{\cancel{2}}(x_n-a)^2 + n\frac{1}{\sigma} = 0$$

$$-\sigma^{-3}(x_1-a)^2 - \dots - \sigma^{-3}(x_n-a) + n\frac{1}{\sigma} = 0$$

$$n\frac{1}{\sigma} = \frac{(x_1-a)^2}{\sigma^3} + \dots + \frac{(x_n-a)^2}{\sigma^3}$$

$$n\frac{1}{\cancel{\sigma}} = \frac{1}{\sigma^3}\left[(x_1-a)^2 + \dots + (x_n-a)^2\right]$$

$$n = \frac{1}{\sigma^2}\left[(x_1-a)^2 + \dots + (x_n-a)^2\right]$$

$$\sigma^2 = \frac{(x_1-a)^2 + \dots + (x_n-a)^2}{n}$$

⚠ You choose something with the highest probability.

- **Data Fusion**: If you have several measurements of a source, you merge them together.

X - Same quantity.
$\tilde{x}_1, \tilde{x}_2$ - two results measuring the same quantity

$\Delta x_1 - \tilde{x}_1 - x_1$ - normally distr w/0 mean and st. dev. σ_1

$\Delta x_2 - \tilde{x}_2 - x$ - normally distr.

$$\tilde{x}_1 = 140 \quad \pm \underbrace{30}_{=\sigma_1}$$
$$\tilde{x}_2 = 110 \quad \pm \underbrace{10}_{=\sigma_2}$$

Common Sense approach
Choose a more accurate value

Ideally do not ignore
Combine

20:50

Idea: Same Idea:

We have two measurements

$$\frac{1}{\sqrt{2\pi}\,\sigma_1}\exp\left(-\frac{(x-\tilde{x}_1)^2}{2\sigma_1^2}\right)\Delta x \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2}\exp\left(\frac{-(x-\tilde{x}_2)^2}{2\sigma_2^2}\right)\Delta x \rightarrow \text{Max}$$

the prob. for the 1st measurement we got x_1 and second we got x_2

⚠ We choose the one with highest probability.

⚠ How to deal with this?

We take the logarithms

We conclude:

$$\frac{-(x-\tilde{x}_1)^2}{2\sigma_1^2} - \frac{(x-\tilde{x}_2)^2}{2\sigma_2^2} \rightarrow \text{Max}$$

When is this the max?

a: When the reverse is the minimum.

this is what is called the least square Methods (LSM)

$$\frac{(x-\tilde{x}_1)^2}{2\sigma_1^2} + \frac{(x-\tilde{x}_2)^2}{2\sigma_2^2} \rightarrow \min$$

$$\frac{2(x-\tilde{x}_1)}{\sigma_1}$$

$$\frac{\cancel{2}(x-\tilde{x}_1)}{\cancel{2}\sigma_1^2} + \frac{\cancel{2}(x-\tilde{x}_2)}{\cancel{2}\sigma_2^2} = 0$$

$$x(\sigma_1^{-2}+\sigma_2^{-2}) = \tilde{x}_1\sigma_1^{-2} + \tilde{x}_2\sigma_2^{2}$$

$$x = \frac{\tilde{x}_1\sigma_1^{-2} + \tilde{x}_2\sigma_2^{2}}{\sigma_1^{-2}+\sigma_2^{2}}$$

Case 1:

$\tilde{x}_1 = 140 \quad \pm 10 = \sigma_1$

$x_2 = 110 \quad \pm 10 = \sigma_2$

$$\frac{110 \cdot \cancel{\frac{1}{100}} + 140 \cdot \cancel{\frac{1}{100}}}{\cancel{\frac{1}{100}} + \cancel{\frac{1}{100}}} = \frac{110 + 140}{2} = \frac{250}{2} = 125$$

Case 2:

$\tilde{x}_1 = 140 \quad \pm 30 = \sigma_1$

$\tilde{x}_2 = 110 \quad \pm 10 = \sigma_2$

$$\frac{110 \cdot \frac{1}{9} + 140}{\frac{1}{9} + 1}$$

$$\frac{140 \cdot \frac{1}{900} + 110 \cdot \frac{1}{100}}{\frac{1}{900} + \frac{1}{100}} =$$

$$\frac{40 \cdot \frac{1}{9} + 110}{\frac{1}{9} + 1} = \frac{140 + 9 \cdot 110}{1 + 9} = \frac{1130}{10} = 113$$

37:06

3/5

Exercise Do the case of n measurment.

$X \approx x_1 \; \sigma_1$

$X \approx x_2 \; \sigma_2$

93:00.

$X \approx x_n \sigma_n$.

$$\boxed{\frac{1}{\sqrt{2\pi}\sigma_1}} \exp\left(-\frac{(x-\tilde{x}_1)^2}{2\sigma_1^2}\right)\Delta x \cdot \ldots \cdot \boxed{\frac{1}{\sqrt{2\pi}\sigma_n}} \exp\left(-\frac{(x-\tilde{x}_n)^2}{2\sigma_n^2}\right)\Delta x \rightarrow \text{Max}$$

$$-\frac{(x-\tilde{x}_1)^2}{2\sigma_1^2} - \ldots - \frac{(x-\tilde{x}_n)^2}{2\sigma_n^2} \longrightarrow \text{Max}$$

Invert.

$$\frac{(x-\tilde{x}_1)^2}{2\sigma_1^2} + \ldots + \frac{(x-x_n)^2}{2\sigma_n^2} \longrightarrow \text{min}.$$

$$\frac{\cancel{2}(x-\tilde{x}_1)}{\cancel{2}\sigma_1^2} + \ldots + \frac{\cancel{2}(x-x_n)}{\cancel{2}\sigma_n^2} = 0$$

$$x\left(\sigma_1^{-2} + \ldots + \sigma_n^{-2}\right) = x_1\sigma_1^2 + \ldots + x_n\sigma_n^2$$

$$x = \frac{\tilde{x}_1\sigma_1^{-2} + \ldots + \tilde{x}_n\sigma_n^{-2}}{\sigma_1^{-2} + \ldots + \sigma_n^{-2}}$$

How accurate is it?

Same constant ... $\cdot \exp\left(-\dfrac{(x-\tilde{x}_1)^2}{2\sigma_1^2} - \dfrac{(x-\tilde{x}_1)^2}{2\sigma_2^2}\right)$

The normal distribution we have it in this type.

If we can transform

$$\text{const} \cdot \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right)$$

$$\frac{(x-\tilde{x}_1)^2}{2\sigma_1^2} + \frac{(x-\tilde{x}_2)^2}{2\sigma_2^2} \stackrel{?}{=} \frac{(x-a)^2}{2\sigma^2}$$

$$\frac{x^2 - 2x\tilde{x}_1 - \tilde{x}_1^2}{2\sigma_1^2} + \frac{x^2 - 2x\tilde{x}_2 - \tilde{x}_2^2}{2\sigma_2^2}$$

$$x^2\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\right) - x\left(\frac{\tilde{x}_1}{\sigma_1^2} + \frac{\tilde{x}_2}{\sigma_2^2}\right) = \left(\frac{\tilde{x}_1^2}{\sigma_1^2} + \frac{\tilde{x}_1^2}{\sigma_2^2}\right.$$

$$\frac{1}{2\sigma^2} = \frac{1}{\cancel{2}\sigma_1^2} + \frac{1}{\cancel{2}\sigma_2^2}$$

$$\sigma^2 = \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{1}{\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

5/3

$$* \; \text{Cosn} \quad \exp\left(-\left(\frac{x-x_1}{2\sigma_1^2}\right)^2 + \dots + \frac{(x-x_n)^2}{2\sigma_n^2} \right)$$

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_1^2} + \dots + \frac{1}{\sigma_n^2}$$

$$\sigma^2 = \frac{1}{\frac{1}{\sigma_1^2} + \dots + \frac{1}{\sigma_n^2}} = \frac{1}{\frac{n}{\sigma_1^2}} = \frac{\sigma_1^2}{n}$$

$\boxed{HW}$

start with some value. x, $\sigma_1, \dots, \sigma_n$ — input

① Simulate measurmet errors

repeat

$$\begin{cases} \tilde{x}_1 = (X + \text{gauss}\dots) \cdot \sigma_1 \\ \vdots \\ \tilde{x}_n = (X + \text{gauss}) \cdot \sigma_n \end{cases}$$

$$\sqrt{\frac{\sum (x_i - a)^2}{n}}$$

$$X_{new} = \frac{\tilde{x}_1 \sigma_1^{-2} + \dots + \tilde{x}_n \sigma_n^{-2}}{\sigma_1^{-2} + \dots + \sigma_n^{-2}}$$

$$\sqrt{\frac{1}{N} \sum (X_{new} - X)^2} \approx \frac{1}{\sigma_1^{-2} + \dots + \sigma_n^{-2}}$$