We have : data fusion.

- several measurements of the same quantity $\tilde{x}_1, \dots \tilde{x}_n$

$\sigma_1, \dots \sigma_n$ - maybe same accuracy.

$$\sigma^2 = \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \cdots \frac{1}{\sigma_n^2}}$$

$$\sigma_1 = \sigma_2 \cdots = \sigma_n$$

$$\downarrow$$

$$\sigma^2 = \frac{\sigma_i^2}{n}$$

$$\sigma = \frac{\sigma_i}{\sqrt{n}}$$

Rule of Thumb:

The accuracy of any statistical estimate based on sample of size $n \simeq \frac{1}{\sqrt{n}}$ in many cases.

Ideal Situation (Data Fusion) $\longrightarrow$
(when we can measure)

many quantities y cannot be measured directly.

$y = f(x_1, x_2 \dots x_n)$ $\Leftarrow$ find some quantities which are easier to measure & related to y.

x_1 - mass

x_2 - volume.

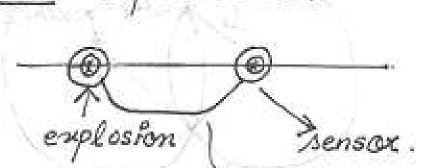
$y = \frac{x_1}{x_2} \Rightarrow f(x_1, x_2)$

- eg: any quantities inside the earth - temp, density
- distance to a star.

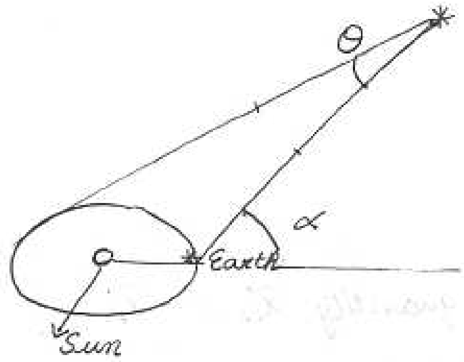
Seismic Experiments:

Depending on density changes, trajectory of sound changes.

- Passive Experiments
- Active Experiments.



by the sound travel, determine the density.



The earth changes positions according to seasons.

Passive Exps:

Indirect Measurement

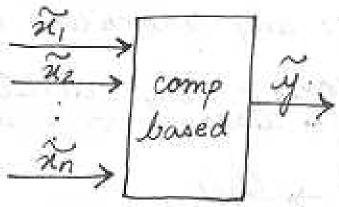
- we measure auxilliary quantities.

  $\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n$

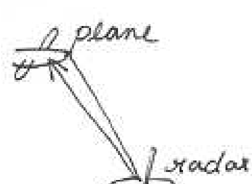
- we plug into f, $\tilde{y} = f(\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n)$.

Additional Problem:

f is not known exactly.

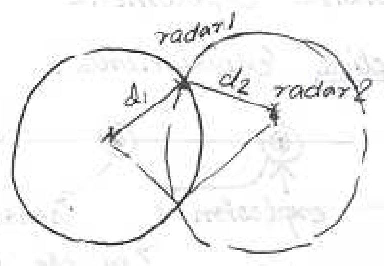


Another eg:

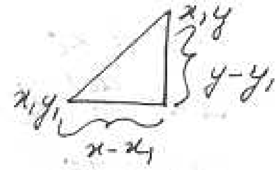


$$t = \frac{2.d}{v}$$

$$d = \frac{vt}{2}.$$



co-ords of first radar = (x_1, y_1)
" " second " = (x_2, y_2)
" " plane = (x, y) - ??known.



$(x_1-x)^2+(y_1-y)^2=d_1^2$ . . . . ①
$(x_2-x)^2+(y_2-y)^2=d_2^2$ . . . . ②

1st Method (complicated).

From ① : $(y_1-y)^2=d_1^2-(x_1-x)^2$

$y_1-y=\sqrt{d_1^2-(x_1-x)^2}$

$y=y_1-\sqrt{d_1^2-(x_1-x)^2}$

In ② $(x_1-x)^2+\left(y_2-y_1-\sqrt{d_1^2-(x_1-x)^2}\right)^2=d_2^2$

2nd Method:

$x_1^2-2xx_1+x^2+y_1^2-2yy_1+y^2=d_1^2$

$-\quad x_2^2-2xx_2+x^2+y_2^2-2yy_2+y^2=d_2^2$

$-2(x_1-x_2)+x_1^2-x_2^2-2(y_1-y_2)+y_1^2-y_2^2=d_1^2-d_2^2$

$2(y_1-y_2)y=d_2^2-d_1^2-2(x_1-x_2)x+x_1^2-x_2^2+\cancel{2(y_1-y_2)y}+y_1^2-y_2^2$

$y=Ax+B$

$A=\dfrac{(x_1-x_2)}{(y_1-y_2)}$, $B=\dfrac{d_2^2-d_1^2+x_1^2-x_2^2+y_1^2-y_2^2}{2(y_1-y_2)}$

We have : $\tilde{y} = f(\tilde{x}_1, \tilde{x}_2, \cdots \tilde{x}_n)$.
We want : $y = f(x_1, x_2, \cdots x_n)$.
Measurements are never absolutely accurate.

$$\tilde{x}_i \neq x_i$$

$$\Delta x_i \stackrel{def}{=} \tilde{x}_i - x_i \neq 0.$$

$d = 6.75.$ ← actual value.

$x_2 = 1.01.$ $\quad \tilde{x}_2 = 1.$

$x_1 = 7.425.$ $\quad \tilde{x}_1 = 7$

$$\begin{array}{r} 6.75 \\ 0.675 \\ \hline 7.425 \end{array}$$

$$\tilde{y} = \frac{7}{1} = 7$$

How do we gauge the difference.

$$\Delta y = \tilde{y} - y?$$

How do we find the difference?

$$\underbrace{x_i}_{\text{unknown}} = \underbrace{\tilde{x}_i}_{\text{known.}} - \Delta x_i \text{ — unknown.}$$

normally distributed.
0 mean.
known standard deviation $\sigma_{\tilde{x}_i}$.

$$\Delta y = \tilde{y}_i - y =$$

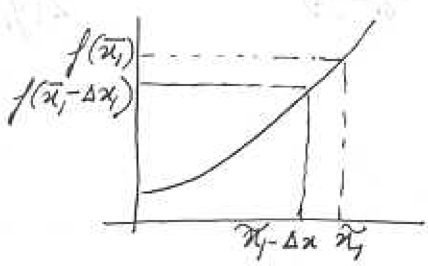
$$f(\tilde{x}_1, \tilde{x}_2 \cdots \tilde{x}_n) - f(x_1, x_2, \cdots x_n)$$

$$= f(\tilde{x}_1, \tilde{x}_2 \cdots \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1 \cdots \tilde{x}_n - \Delta x_n)$$

good news : Δx_i are small.

So $f(\tilde{x}_1) - f(\tilde{x}_1 - \Delta x_1) \simeq \Delta y.$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\partial f}{\partial x_1}$$



$$\therefore \Delta y \approx \frac{\partial f}{\partial x_1} \cdot \Delta x.$$

$f(x_1, x_2 \cdots x_n)$

$\tilde{x}_1, \tilde{x}_2 \cdots \tilde{x}_n$ - measured values.

$\tilde{y} = f(\tilde{x}_1, \tilde{x}_2 \cdots \tilde{x}_n)$ - result of indirect measurement.

estimate.

$\Delta y = \tilde{y} - y.$

$\Delta x_i = \tilde{x}_i - x_i$ - error of direct measurement.

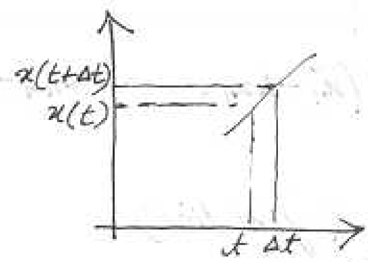
$x_i = \tilde{x}_i - \Delta x_i$

$\Delta y = f(\tilde{x}_1, \cdots \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \cdots \tilde{x}_n - \Delta x_n).$

$$\frac{F(a+\Delta a) - F(a)}{\Delta a} \approx F'(a)$$

$F(a+\Delta a) - F(a) \approx F'(a) . \Delta a.$

$F(a+\Delta a) \approx F'(a) . \Delta a + F(a)$



$$v = \frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

We have, $\Delta y = f(x_1) - f(\tilde{x}_1 - \Delta x_1)$

$f(\tilde{x}_1 - \Delta x_1) = f.$

$\Delta a = -\Delta x_1.$

$$f(\tilde{x}_1 - \Delta x_1) = f(\tilde{x}_1 + (-\Delta x_1)) = f(\tilde{x}_1) + \frac{df}{dx_1}(-\Delta x_1).$$

$$= f(\tilde{x}_1) - \frac{df}{dx_1}(\Delta x_1)$$

$$f(\tilde{x}_1) - \left(f(\tilde{x}_1) - \frac{df}{dx}(\Delta x_1)\right) = \cancel{f(\tilde{x}_1)} - \cancel{f(\tilde{x}_1)} + \frac{df}{dx}(\Delta x_1)$$

For, $n = 1$ $\quad \Delta y = \frac{df}{dx} \Delta x_1$

$f = x_1 . x_2^2$

$\frac{\partial f}{\partial x_1} = x_2^2$, $\frac{\partial f}{\partial x_2} = 2 . x_1 x_2$

$f(\tilde{x}_1, \cdots \tilde{x}_n)$

$f(\tilde{x}_1 - \Delta x_1, \cdots \quad \tilde{x}_n - \Delta x_n)$.

$$\Delta y = \big(f(\tilde{x}_1, \tilde{x}_2, \cdots \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \tilde{x}_2, \cdots \tilde{x}_n)\big) +$$
$$\big(f(\tilde{x}_1 - \Delta x_1, \tilde{x}_2 \cdots \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \tilde{x}_2 - \Delta x_2 \cdots \tilde{x}_n)\big) +$$
$$\big(f(\tilde{x}_1 - \Delta x_1, \tilde{x}_2 - \Delta x_2 \cdots x_n) - f(\tilde{x}_1 - \Delta x_1, \tilde{x}_2 - \Delta x_2, \tilde{x}_3 - \Delta x_3 \cdots \tilde{x}_n)\big) +$$
$$\vdots$$
$$\big(f(\tilde{x}_1 - \Delta x_1, \cdots x_{n-1} - \Delta x_{n-1}, x_n) - f(\tilde{x}_1 - \Delta x_1, x \cdots, \tilde{x}_n - \Delta x_n)\big)$$

$\frac{\partial f}{\partial x_1} \cdot \Delta x_1$ (first line)

$\frac{\partial f}{\partial x_2} \cdot \Delta x_2$ (second line)

$\vdots$

$\frac{\partial f}{\partial x_n} \cdot \Delta x_n$ (last line)

$$\therefore \Delta y = \left(\frac{\partial f}{\partial x_1} \cdot \Delta x_1 + \frac{\partial f}{\partial x_2} \cdot \Delta x_2 + \cdots \frac{\partial f}{\partial x_n} \cdot \Delta x_n\right) \quad \text{linearization}$$
$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i$$

Ohms Law (eg)

$V = IR.$ $\quad \therefore \dfrac{\partial f}{\partial x_1} = x_2.$

$\uparrow\ \uparrow$

$x_1\ x_2$

$f(x_1, x_2) = x_1 . x_2.$ $\quad \therefore \dfrac{\partial f}{\partial x_2} = x_1$

$x_1 = 1.0,\ x_2 = 2.0,\ \Delta x_1 = +0.1,\ \Delta x_2 = -0.1$

$x_1 = 1.0 - 0.1 = 0.9$

$x_2 = 2.0 + 0.1 = 2.1$

$\therefore \tilde{y} = \tilde{x}_1 . \tilde{x}_2 = 1.0 * 2.0 = 2.0$

$y = x_1 . x_2 = 0.9 . 2.1 = 1.89.$

$\therefore \Delta y = 2.0 - 1.89 = 0.11$

<u>using</u> <u>linearization</u>:

$\dfrac{\partial f}{\partial x_1} = 2.0 \quad , \quad \dfrac{\partial f}{\partial x_2} = 1.0$

$\Delta y = 2.0 . \Delta x_1 + 1.0 . \Delta x_2$

$= 2.0 . 0.1 + 1.0 \times (-0.1)$

$= 2.0 - 0.1 = 0.1$

$\therefore$

<u>Eg 2</u>: $y = x_1 x_2^2.$

$\ \ \shortparallel \quad \shortparallel \quad \shortparallel$

$\ \ P. \quad R \quad I$

$x_1 = 0.9$ $\quad \dfrac{\partial f}{\partial x_1} =$

$x_2 = 2.1$

$\therefore f(x_1, x_2) = x_1 x_2^2 \quad \dfrac{\partial f}{\partial x_1} = x_2^2. \quad \dfrac{\partial f}{\partial x_2} = 2x_1 x_2.$

$\therefore \cancel{x_1} = \Delta\tilde{y} = x_1 . x_2^2 = 1.0 \times 4.0 \quad \dfrac{\partial f}{\partial x_2} = 4.0$

$y = 0.9 \times (2.1)^2 = 3.969$

$\Delta y = 4.0 - 3.969 = 0.031$

$\Delta y_l = 4.0 . 0.1 + 2.1.0 \times (-2.0)$

$= 0.4 + (-0.4) = 0$

$2.1 \times 2.1 = 4.41$

$4.000 - 3.969 = .031$

H/W

① By hand,

$f = (x_1^2 + x_2^2)$

$\tilde{x}_1 = 1.0, \quad \tilde{x}_2 = 2.0, \quad \Delta x_1 = +0.1, \quad \Delta x_2 = -0.1.$

② Using computer.

$y = \frac{x_1}{x_2}$, $\tilde{x}_1 \; \tilde{x}_2 \; \Delta x_1 \; \Delta x_2$.

We know,

$\Delta x_1 \ldots \Delta x_n \Rightarrow$ 0 mean and know SD $\sigma_1, \ldots \sigma_n$.

we want to find st, mean of Δy:

Simple case: $n = 1$.

$$\Delta y = \frac{df}{dx_1} \cdot \Delta x_1.$$

$$E[a] = \text{Mean} = \lim_{N \to \infty} \frac{a^{(1)} + \ldots \ldots a^{(N)}}{N}$$

$$E[K.a] = \frac{K.a^{(1)} + \ldots + K.a^{(N)}}{N} = K.\left(\frac{a^{(1)} + \ldots \ldots a^{(N)}}{N}\right) = K.\left(\frac{a^{(1)} + a^{(2)} + \ldots a^{(N)}}{N}\right)$$

If we multiply a random variable by a constant the mean is ~~the~~ also multiplied by the constant.

Standard Deviation: (SD) =

$$\sigma[a] = \sqrt{\frac{(a^{(1)} - E([a]))^2 + (a^{(2)} - E([a]))^2 + \ldots \ldots (a^{(N)} - E([a]))^2}{N}}$$

$$\sigma[K.a] = \sqrt{\frac{(K.a^{(1)} - K.E[a])^2 + (K.a^{(2)} - K.E[a])^2 + \ldots (K.a^{(N)} - K.E[a])^2}{N}}$$

$$= \sqrt{\frac{K^2 (a^{(1)} - E[a])^2 + K^2 (a^{(2)} - E[a])^2 + \ldots K^2 (a^{(N)} - E[a])^2}{N}}$$

$$= \sqrt{\frac{K^2(a^{(1)} - E[a])^2 + \cdots + (a^{(N)} - E[a])^2}{N}}$$

$E[K.a] = K.E[a]$.

$\sigma[K.a] = |K| . \sigma[a]$

$E[\Delta x_i] = 0$

$\sigma[\Delta x_1] = \sigma_1$

$\Delta y = \frac{df}{dx_1} . \Delta x_1$

$E[\Delta y] = 0$

$\sigma[\Delta y] = \left|\frac{df}{dx_1}\right| \times \sigma_1 = \sqrt{\left(\frac{df}{dx}\right)^2 \cdot \sigma_1^2}$

---

a b

$E[a]$ $E[b]$

$E[a+b] \stackrel{?}{=} E[a] + E[b]$

Proof :

$$E[a+b] = \frac{(a^{(1)} + b^{(1)}) + \cdots + (a^{(N)} + b^{(N)})}{N}$$

$$= \frac{a^{(1)} + b^{(1)} + \cdots + a^{(N)} + b^{(N)}}{N} = \frac{a^{(1)} + a^{(2)} + \cdots a^{(N)}}{N} + \frac{b^{(1)} + b^{(2)} + \cdots b^{(N)}}{N}$$

$= E[a] + E[b]$.

In case of SD, the variables may be independent.