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We have a function: $f(x_1, \ldots, x_n)$

measured values : $\tilde{x}_1, \ldots, \tilde{x}_n$

result of indirect measurement : $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$

estimate

$$\Delta y = \tilde{y} - y$$

$\Delta x_i = \tilde{x}_i - x_i$ - error of direct measurement

$$x_i = \tilde{x}_i - \Delta x_i$$

$$\Delta y = f(\tilde{x}_1, \ldots, \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \ldots, \tilde{x}_n - \Delta x_n)$$

$$\frac{f(a + \Delta a) - f(a)}{\Delta a} \approx f'(a)$$

$$f(a + \Delta a) - f(a) \approx f'(a)\,\Delta a$$

$$f(a + \Delta a) \approx f'(a)\Delta a + f(a)$$

$$\Delta y = f(\tilde{x}_1) - \underbrace{f(\tilde{x}_1 - \Delta x_1)}$$

7:54 $\quad f(\tilde{x}_1 - \underline{\Delta x_1}) \approx f \qquad a = \quad f(\tilde{x}_1) - \left(f(\tilde{x}_1) - \frac{\partial f}{\partial x}\Delta x_1\right) =$

$$\Delta a = -\Delta x_1 \qquad = \cancel{f(\tilde{x})} - \cancel{f(\tilde{x})} + \frac{\partial f}{\partial x}\Delta x_1$$

$$\boxed{n = 1 \quad \Delta y = \frac{\partial f}{\partial x_1}\Delta x_1}$$

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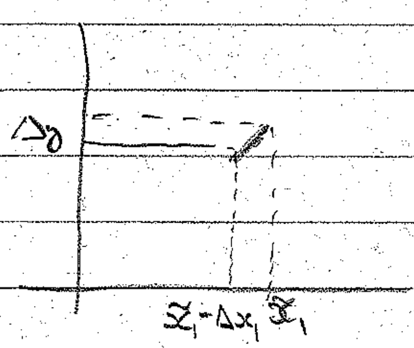
$$f(\tilde{x}_1 - \Delta x_1) =$$

$$f(\tilde{x}_1 + (-\Delta x_1)) = f(\tilde{x}_1) + \frac{\partial f}{\partial x_1}(-\Delta_1) =$$

$$f(\tilde{x}_1) - \frac{\partial f}{\partial x_1}\Delta x_1$$



⚠ Several measurements

$f = x_1 x_2^2$

- $\frac{\partial f}{\partial x_1} = x_2^2$ ← How does this relate?
- $\frac{\partial f}{\partial x_2} = x_1 \cdot 2x_2$

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If we have:

$f(\tilde{x}_1, \ldots, \tilde{x}_n)$

$f(\tilde{x}_1 - \Delta x_1, \ldots, \tilde{x}_n - \Delta x_n)$

We need to come up with simpler expression.

$\frac{\partial f}{\partial x_1} \cdot \Delta x_1$

In the process We already know how to evaluate. So we know the beginning and the end

$$
\begin{aligned}
&① \ \Delta y = \left(f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \tilde{x}_2, \ldots, \tilde{x}_n)\right) + \\
&② \quad \left(f(\tilde{x}_1 - \Delta x_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \tilde{x}_2 - \Delta x_2, \ldots, \tilde{x}_n)\right) + \\
&\quad \vdots \\
&ⓝ \quad \left(f(\tilde{x}_1 - \Delta x_1, \ldots, \tilde{x}_{n-1} - \Delta x_{n-1}, \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \ldots, \tilde{x}_n - \Delta x_n)\right)
\end{aligned}
$$

For the first line ① $\longleftrightarrow$ $\frac{\partial f}{\partial x_1} \cdot \Delta x_1$

" 2^{nd} line ② $\longleftrightarrow$ $\frac{\partial f}{\partial x_2} \cdot \Delta x_2$

$\vdots$

n^{th} " ⓝ $\longleftrightarrow$ $\frac{\partial f}{\partial x_n} \cdot \Delta x_n$

This is called $\boxed{\text{linearization.}}$

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$$\boxed{\Delta y = \frac{\partial f}{\partial x_1} \cdot \Delta x_1 + \ldots + \frac{\partial f}{\partial x_n} \cdot \Delta x_n}$$

Some example:

$CV = I R_w$

$y = x_1 \; x_2$

$f(x_1, \ldots, x_2) = x_1 \cdot x_2$

$\frac{\partial f}{\partial x_1} = x_2$

$\frac{\partial f}{\partial x_2} = x_1$

Let's pick some values

$\tilde{x}_1 = 1.0 \qquad \tilde{x}_2 = 2.0$

$\Delta x_1 = \pm 0.1 \qquad \Delta x_2 = -0.1$

$X_1 = 0.9 \qquad X_2 = 2.0 - (-0.1) = 2.1$

$\tilde{y} = \tilde{x}_1 \cdot \tilde{x}_2 = 1.0 \cdot 2.0 = 2.0$

$y = x_1 \cdot x_2 = (0.9)(2.1) = 1.89$

$\Delta y = 2.0 - 1.89 = \underline{\underline{0.11}}$

$$\boxed{\begin{gathered} \frac{\partial f}{\partial x_1} = 2.0 \qquad \frac{\partial f}{\partial x_2} = 1.0 \\ \Delta y = 2.0 \, \Delta x_1 + 1.0 \, \Delta x_2 = \\ 2.0 \cdot (0.1) + 1.0 \cdot (-0.1) = \\ 0.2 - 0.1 = 0.1 \end{gathered}}$$

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Numeric example

$y = x_1 x_2^2$ $\quad \tilde{x}_1 = 1.0 \quad \tilde{x}_2 = 2.0$

$\Delta x_1 = +0.1 \quad \Delta x_2 = -0.1$

$\frac{\partial f}{\partial x_1} = x_2^2$

$\frac{\partial f}{\partial x_2} = x_1 2 x_2$

$\tilde{y} = \tilde{x}_1 \tilde{x}_2^2 = 1.0 \cdot (2.0)^2 = 4.0$

$y = x x_2^2 = (0.9) \cdot (2.1)^2 = (0.9)(4.41) = 3.969$

$\Delta y = \tilde{y} - y$

$\Delta y = 4.0 - 3.969 = \underline{0.031}$

$\frac{\partial f}{\partial x_1} = (2.0)^2 \qquad \frac{\partial f}{\partial x_2} = 2(0.9)(2.1)^2$

$\Delta y = \quad \Delta x_1 + \quad \Delta_2$

HW 1st part

① By hand $F = x_1^2 + x_2^2$

$\tilde{x}_1 = 1.0 \quad \tilde{x}_2 = 2.0$

$\Delta x_1 = +0.1 \quad \Delta x_2 = -0.1$

② computer $y = \frac{x_1}{x_2}$

$\tilde{x}_1, \tilde{x}_2 \quad \Delta x_1 \quad \Delta x_2$

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Suppose now that we have mean & stand deviation

$$\Delta y = f(\tilde{x}_1, \dots, \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \dots, \tilde{x}_n - \Delta x_n) \approx \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i$$

We know $\begin{cases} \Delta x_1, \dots, \Delta x_n \\ 0 \text{ mean and known st. deviations } \sigma_1, \dots, \sigma_n \end{cases}$

We want { We want to find: st. deviation of Δy

Simple case $\boxed{n=1}$

$$\Delta y = \frac{\partial f}{\partial x_1} \Delta x_1$$

$$E[a] = \lim_{N \to \infty} \frac{a^{(1)} + \dots + a^{(N)}}{N}$$

$$E[Ka] = \frac{Ka^{(1)} + \dots + Ka^{(N)}}{N} = \frac{K(a^{(1)} + \dots + a^{(N)})}{N} = \boxed{K\left(\frac{a^{(1)} + \dots + a^{(N)}}{N}\right)}$$

Let's look the stn deviation.

$$\sigma[a] = \sqrt{\frac{(a^{(1)} + E[a])^2 + (a^{(2)} + E[a])^2 + \dots + (a^{(N)} + E[a])^2}{N}}$$

Let's assume we multiply by a constant.

$$\sigma[Ka] = \sqrt{\frac{(K a^{(1)} - K \cdot E[a])^2 + \dots + (K \cdot a^{(N)} - K E[a])^2}{N}}$$

$$= \sqrt{\frac{K^2(a^{(1)} - E[a])^2 + \dots + K^2(a^{(N)} - E[a])^2}{N}}$$

$$= \sqrt{\frac{K^2[(a^{(1)} - E[a])^2 + \dots + (a^N - E[a])^2]}{N}}$$

$$= |K| \cdot \sigma[a]$$

If we have 1 variable.

$E[Ka] = K\ E[a]$

$\sigma[Ka] = |K| \sigma[a]$

$E[\Delta x_1] = 0$

$\sigma[\Delta x_1] = \sigma_1$

$\Delta y = \frac{\partial f}{\partial x_1} \cdot \Delta x_1$

$E[\Delta y] = 0$

$$\sigma[\Delta y] = \left|\frac{\partial f}{\partial x_1}\right| \cdot \sigma_1 = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \cdot \sigma_1^2}$$

$$\overset{a}{E[a]} \qquad \overset{b}{E[b]}$$

$$E[a+b] \stackrel{?}{=} E[a] + E[b]$$

Let's prove it

$$E[a+b] = \frac{(a^{(1)}+b^{(1)}) + \ldots + (a^{(N)}+b^{(N)})}{N} =$$

$$= \frac{a^{(1)}+b^{(1)}+ \ldots + a^{N}+b^{(N)}}{N} = \frac{(a^{(1)}+\ldots+a^{N})+(b^{(1)}+\ldots+b^{(N)})}{N}$$

$$= \underbrace{\frac{a^{(1)}+\ldots+a^{(N)}}{N}}_{= E[a]} + \underbrace{\frac{b^{(1)}+\ldots+b^{(N)}}{N}}_{= E[b]}$$

What does it mean independent?

$$E[f(a)\cdot g(b)] = E[f(a)] \cdot E[g(b)]$$