$$E[f(a)g(b)] = E(f(a)) \cdot E(g(b)).$$

$$\Delta y = \frac{\partial f}{\partial x_1} \Delta x_1 + \frac{\partial f}{\partial x_2} \cdot \Delta x_2 + \cdots\cdots + \frac{\partial f}{\partial x_n} \cdot \Delta x_n .$$

1) What is independence? Why it is related to the product?

<u>Fact</u>: A and B are 2 independent events.

$$P(A \& B) = P(A).P(B)$$

<u>Why</u>:

<u>Definition of Probability</u>:

$P(A) = \frac{N(A)}{N}$ # no of cases when A occurred. / # of cases.

A - raining in El Paso.

$N = 10,000$ records. $\therefore P(A) \approx 0.1024$

$N(A) = 1024$ days.

What if we only consider some records?

① only records with lightening, $N \approx 950$.

$N(A) \approx$ No of days with lightening ≈ 450

$$P(A) \approx \frac{N(A \& B)}{N(B)} \approx \frac{450}{950} \approx 0.5 \neq 0.1024.$$

Because they are not independent.

(2) only records made on even days.

$N(B) \approx 5{,}000$.

$N(A \& B) \approx 500$

$\frac{N(A\&B)}{N(B)} \approx \frac{N(A)}{N}$

independence means that the portions of those who satisfy A does not change if we limit ourselves to those who have B.

$N(A)$ - total # of cases when A is true

N - total # of records.

$N(B)$ - total # of cases when B is true

$N(A\&B)$ - . . . . A&B . . . .

Ans.

$\frac{N(A\&B)}{N} \approx P(A\&B)$

$N(A\&B) \approx N \cdot P(A\&B)$

$\frac{\cancel{N} \cdot P(A\&B)}{\cancel{N} \cdot P(B)} \approx \frac{\cancel{N} \cdot P(A)}{\cancel{N}}$

$[N(A) = N \cdot P(A)]$

$\frac{P(A\&B)}{P(B)} = P(A)$.

$\boxed{P(A\&B) = P(A) \cdot P(B)}$

[conditional probability]

Q2) What is the expected (mean) value of a random variable?

Def1: take several samples $a^{(1)}, a^{(2)} \ldots\ldots a^{(n)}$.

$$E[a] \approx \frac{a^{(1)} + a^{(2)} + \ldots\ldots + a^{(n)}}{n}.$$

Def2: $E[a] = \sum_{i=1}^{n} p_i a_i$

How are defn 1 and defn 2. related.

Lets assume that a has n possible values,

$$a_1, a_2 \cdots\cdots a_n.$$

$$\frac{a^{(1)}+a^{(2)}+\cdots a^{(n)}}{n} = \frac{a_1 n(a_1) + a_2 \cdot n(a_2) + \cdots a_n n(a_n)}{n}$$

prob of a_1 is p_1 $\sum_{i=1}^{n} p_i = 1$

prob of a_2 is p_2

$\vdots$

prob of a_n is p_n $\boxed{p_i = \frac{N(a_i)}{N}}$

$p_1 = p(a_1) = \frac{N(a_1)}{N}$

eg: $$\frac{\overset{a(1)}{3}+\overset{a(2)}{4}+\cancel{3}+\overset{a(3)}{3}+\overset{a(4)}{4}+\overset{a(5)}{3}+\overset{a(6)}{4}+\overset{a(7)}{5}}{7} = \frac{\overset{a_1 n(a_1)}{3.3}+\overset{a_2 n(a_2)}{4.3}+\overset{a_3 n(a_3)}{5.1}}{7}$$

$$= \frac{a_1 . n(a_1)}{n} + \frac{a_2 . n(a_2)}{n} + \cdots + \frac{a_n . n(a_n)}{n}$$

$$\approx p_1 . a_1 + p_2 . a_2 + \cdots + p_n . a_n.$$

Expected value of a product:

Two random variables a, b are independent.

a	a_1	a_2	$\cdots$	a_n.
	$p(a_1)$	$p(a_2)$	$\cdots$	$p(a_n)$
b	b_1	b_2	$\cdots$	b_m
	$p(b_1)$	$p(b_2)$	$\cdots$	$p(b_m)$

$$p(a_i \,\&\, b_j) = p(a_i) . p(b_j)$$

$$E[a.b] = \sum_{i=1}^{n} \sum_{j=1}^{m} p(a_i \,\&\, b_j) . p(a_i) p(b_j)$$

$a_i \, b_j$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} p_a(a_i) . p_b(b_j) . a_i . b_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \overbrace{p_a(a_i) \cdot a_i}\ \overbrace{p_b(b_j) \cdot b_j}$$

$$= \sum_{i=1}^{m} p_a(a_i) \cdot a_i \left(\sum_{j=1}^{m} p_b(b_i)\, b_j \right)$$

$$= \underbrace{\left(\sum_{j=1}^{m} p_b(b_j) \cdot b_j \right)}_{E[b]} \underbrace{\left(\sum_{i=1}^{n} p_a(a_i) \cdot a_i \right)}_{E[a].}$$

$s = v.t.$

$s_{avg} = v_{avg} \cdot t_{avg}.$

$$\sigma[a] = \sqrt{\underbrace{\frac{(a^{(1)} - E[a])^2 + \cdots (a^{(N)} - E[a])^2}{N}}}$$

$$\sigma[a) = \quad E(a - E[a])^2 = V(a) = \text{variance}.$$

$$\sigma\, \Delta y = \frac{\partial f}{\partial x_1} \cdot \Delta x_1 + \cdots + \frac{\partial f}{\partial x_n} \cdot \Delta x_n .$$

$$E[\Delta x_i] = 0$$

$$\sigma[\Delta x_i] = \sigma_i$$

Finding S.D of Δy ;

$$V[\Delta y] = E[\Delta y^2]$$

$$= E[(c_1 \Delta x_1 + c_2 \Delta x_2 + \cdots c_n \Delta x_n)^2]$$

$$= E[c_1^2 \Delta x_1^2 + c_2^2 \Delta x_2^2 + \cdots c_n^2 \Delta x_n^2 + 2 c_i c_j \Delta x_i \Delta x_j + \cdots]$$

$$\sigma^2 = c_1^2 \sigma_1^2 + c_2 \cdot \sigma_2^2 + \cdots + c_n \sigma_n^2 .$$

HW:

Summarise:

$f(x_1, x_2 \dots x_n)$ - we know the function.

$\tilde{x}_1, \tilde{x}_2 \dots \tilde{x}_n$ - approximate values.

$\Delta x_i = \tilde{x}_i - x_i$, $E[\Delta x_i] = 0$, we know $\sigma[\Delta x_i]$

$\tilde{y} \stackrel{def}{=} f(\tilde{x}_1, \quad \tilde{x}_n)$ - we compute

$y = \bar{y} - y$.

$y \stackrel{def}{=} f(\tilde{x}_1, x_2 \quad x_n)$

$\hookrightarrow$ actual values of x_i

$E[\Delta y] = 0$

$\sigma[\Delta y] = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2}$

Algorithmically:

Case 1: we know $f(x_1, x_2 \quad x_n)$

analytical expression.

problem: we don't have the expression.

so its not easy to compute the derivative.

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{df}{dx} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ numerical differentiation

$$\frac{\partial f}{\partial x} = \frac{f(x_1, x_2 \dots x_{i-1}, x_i + h_i, \tilde{x}_{i+1}, \dots \tilde{x}_n) - \tilde{y}}{h_i}$$

Given: $f(x_1, \cdots x_n)$

$\tilde{x}_1, \cdots \tilde{x}_n$

$\sigma_1, \sigma_2 \cdot \sigma_n$ $\qquad \tilde{y} = f(\tilde{x}_1, \tilde{x}_2 \cdots \tilde{x}_n)$

$$\frac{\partial f}{\partial x_i} = \frac{f(\tilde{x}_1, \tilde{x}_2 \cdots \tilde{x}_{i-1}, \tilde{x}_i + h_i, \tilde{x}_{i+1} \cdots\cdots \tilde{x}_n) - \tilde{y}}{h_i}$$

$$\sigma = \sqrt{\sum \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2}$$

$$\sigma = \sqrt{\left(\frac{f(\tilde{x}_1 + h, \tilde{x}_2, \cdots \tilde{x}_n) - \tilde{y}}{h_1^2}\right)^2 \cdot \sigma_1^2 + f\left(\frac{x_1, \tilde{x}_2 + h, \cdots \tilde{x}_n) - \tilde{y}}{h_2^2}\right)^2 + \cdots}$$

Trick: $h_i = \sigma_i$

$$\sigma = \sqrt{(f(\tilde{x}_1 + \sigma_1, \tilde{x}_2, \cdots) - \tilde{y})^2 + (f(\tilde{x}_1, \tilde{x}_2 + \sigma_2, \tilde{x}_3 \cdots \tilde{x}_n) - \tilde{y})^2 + \cdots}$$

Homework

1) write a code that computes σ.
2) check your code by comparing with simulation results.

$f(x_1, \cdots x_n)$ – given a function.

$\tilde{x}_1, \tilde{x}_2 \cdots \tilde{x}_n$ – values

$\sigma_1, \sigma_2 \cdots \sigma_n$ – values

sim. $\Delta x_i = \sigma_i \cdot$ gauss()

sim. $x_i = \tilde{x}_i - \Delta x_i$

sim $y = f(x_1, x_2 \cdots x_n)$

sim $\Delta y = \tilde{y} - y$

$$\sigma = \sqrt{\frac{1}{N} \sum_{k=1}^{N} (\Delta y^{(k)})^2}$$