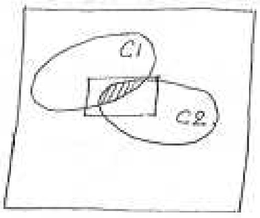
Solving contraints → NP hard problem (using heuristics the time is reduced)



Interval Computations

known relation

So far: $x_1, \dots x_n$; $y = f(x_1, x_2 \dots x_n)$

$\tilde{x}_1, \tilde{x}_2 \dots \tilde{x}_n$ - measurement results

$\tilde{y} = f(\tilde{x}_1, \tilde{x}_2 \dots \tilde{x}_n)$ - result of data processing.

We know:

$$\Delta x_i = \tilde{x}_i - x_i \neq 0$$

So far: assuming that $\Delta x_1, \dots \Delta x_n$ are normally distributed, w/o mean and SD is σ_i (known)

Δy ? → normally distributed, w/o mean and st. dev

$$\sigma = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2}$$

In practice, often we dont know the distribution; all we know is upper bound on measurement error

$\Delta_1, \dots \Delta n$. $\quad |\Delta x_i| \leq \Delta_i$

$x_i \in [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$ → all we know.

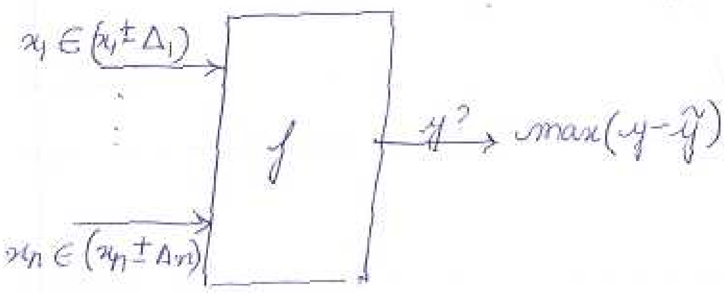
$\tilde{x}_1 = 0.9$, $\Delta_1 = 0.1$

$0.9 + 1 = 1.0$

$0.9 - 0.1 = 0.8$ $\quad \tilde{x}_1 = 0.9$ $\qquad [\tilde{x}_1 - \Delta x_1, \tilde{x}_1 + \Delta_1]$

$\Delta_1 \; \Delta_1$

$x_1 - \Delta_1 \qquad x_1 + \Delta_1$



Using Ohm's Law,

$V = IR$

$\tilde{I} = 1.0,\ \Delta_I = 0.1$ $\qquad I \in [0.9, 1.1]$

$\tilde{R} = 2.0,\ \Delta_R = 0.2$ $\qquad R \in [1.8, 2.2]$

$V = 1.2 = 2.0$

$V = 0.9 \cdot 2.2 = 1.98$ $\qquad [V] = [0.9 \cdot 1.8,\ 1.1 \times 2.2] = [1.62, 2.42]$

$V = 1.1 \times 1.8 = 1.98$

$V = 1.05 \times 2.02 = 2.1$

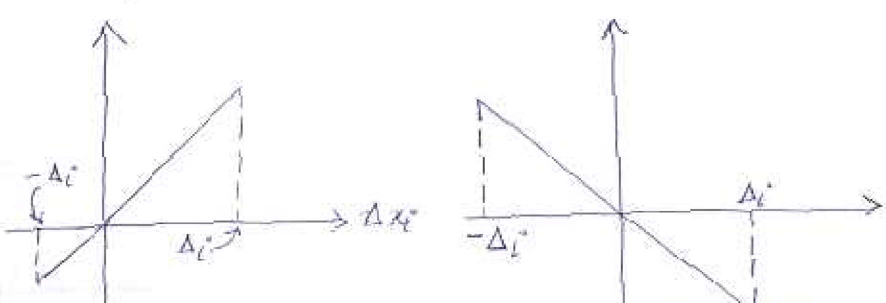
$$\Delta y = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \Delta x_i$$

$\Delta x_i \in [-\Delta_i, \Delta_i]$

when is Δ_i the largest?

<u>inputs</u>: $\Delta_1, \Delta_2 \cdots \Delta_n$

<u>property</u>: Δy is monotonic in each Δx_1



$\frac{\partial f}{\partial x_i} > 0 \Rightarrow \frac{\partial f}{\partial x_i} \cdot \Delta_i$ $\qquad \frac{\partial f}{\partial x_i} < 0 \Rightarrow \frac{\partial f}{\partial x_i}(-\Delta_i) = \left(-\frac{\partial f}{\partial x_i}\right) \Delta_i$

$\left|\frac{\partial f}{\partial x_i}\right|$

conclusion : $\Delta = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| \Delta_i$

$\tilde{I} = 1.0, \ \Delta_I = 0.1.$

$\tilde{R} = 2.0, \ \Delta_R = 0.2.$

$f(I, R) = I.R.$

$\tilde{x}_1 = 1.0, \ \Delta_1 = 0.1.$

$\tilde{x}_2 = 2.0, \ \Delta x_2 = 0.2.$

$f(x_1, x_2) = x_1 . x_2$

$\tilde{y} = 2.$

$[1.62, 2.42]$

actual range.

$\frac{\partial f}{\partial x_1} = x_2 = 2.0$

$\frac{\partial f}{\partial x_2} = x_1 = 1.0$

$\Delta = |2.0| \times 0.1 + |1.0| \times 0.2 = 0.4.$

$y \in [\underset{2.0}{\tilde{y}} - \Delta, \ \tilde{y} + \underset{0.4}{\Delta}]$

$[1.6, 2.4]$

h/w:

1) run a numerical eg of interval computation ; compare with linearized range.

2) write a code to compute Δ by using numerical diff (by hand) & check by exhaustive search.

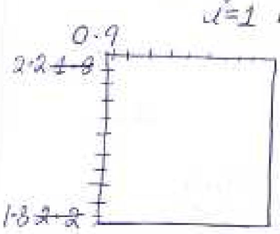
Numerical Differentiation:

$$\frac{\partial f}{\partial x_i} = \frac{f(\tilde{x}_1, \dots \tilde{x}_{i-1}, \tilde{x}_i + h_i, \tilde{x}_{i+1} \dots \tilde{x}_n) - \tilde{y}}{h_i}$$

$$\Delta = \sum_{i=1}^{n} \left| \frac{f(\tilde{x}_1, \dots \tilde{x}_{i-1}, \tilde{x}_i + h_i, \tilde{x}_{i+1} \dots \tilde{x}_n) - \tilde{y}}{h_i} \right| \Delta_i$$

If we choose to simplify computations, we choose $h_i = \Delta_i$.

$$\Delta = \sum_{i=1}^{n} \left| f(\tilde{x}_1, \dots \tilde{x}_{i-1}, \tilde{x}_i + \Delta_i, \tilde{x}_{i+1} \dots \tilde{x}_n) - \tilde{y} \right|$$

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for (x1 = x1 - Δ1; x1 <= x1 + Δ1; x1+ = Δ1/10)
   for (x2 =                              )
        y = f(x1, x2);
        if y > yrec then yrec = y;
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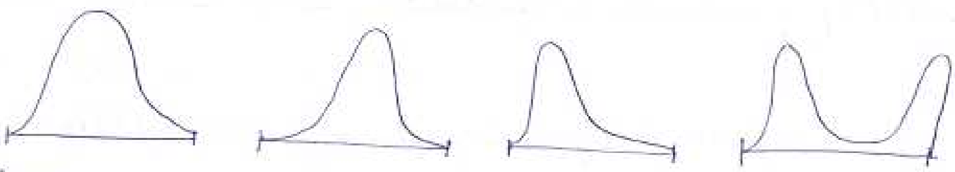
Problems:

① (same as stat.)
   when n is large, we have too many calls to f.

② linerization is an approximate tool. (In general linearization - NP hard)



Ideally : we need $\approx$ Monte Carlo method.

Problem: $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i$

$$\downarrow$$

$$\sigma^2 = \sqrt{\sum \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2}$$

Example : $f(x_1 + x_2 + \cdots x_n) = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n$.

$$|\Delta x_i| \leqslant \Delta_i = \Delta_1 = \Delta_2 = \Delta_3 \cdots = \Delta_n$$

correct answer : $n \cdot \Delta_1$.

$$f(x_1, x_2 \cdots x_n) = x_1 + x_2 + \cdots + x_n$$

$$\tilde{y} = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n$$

$$\Delta y = \tilde{y} - y = (\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n) - (x_1 + x_2 + \cdots + x_n)$$

$$= \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n - x_1 - x_2 - \cdots x_n$$

$$= (\tilde{x}_1 - x_1) + (\tilde{x}_2 - x_2) + \cdots + (\tilde{x}_n - x_n)$$

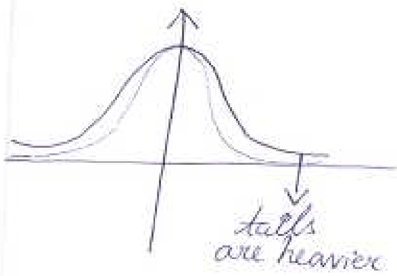
$$= \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n.$$

$$\Delta y = \sum \frac{\partial f}{\partial x_i} \cdot \Delta x_i$$

$\therefore \sqrt{\sum \sigma_i^2} \leqslant \sqrt{n \cdot \Delta_i^2} = \Delta_i \sqrt{n}$ $\left[\text{From } \sigma^2 = \sqrt{\sum \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2}\right]$

$$\sigma_i^2 \leqslant \Delta_i^2$$

Cauchy Distribution:

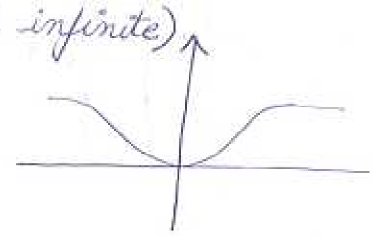


$$\rho(x) = \frac{\Delta}{\pi} \frac{1}{1+(x/\Delta)^2}$$

$$\rho_{Gauss}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad x = 3\sigma$$

In Gaussian distribution; the tails go to 0. In Cauchy's dis; tails are infinite, they never go to 0. (variance is infinite)

$$V = \sum p_i x_i^2 = \int_{-\infty}^{\infty} \rho(x) \cdot x^2 \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)} \, dx.$$



Gaussian Distribution:

$$\Delta y = \sum_{i=1}^{m} c_i \Delta x_i$$

Δx_i are Gaussian st dev σ_i then Δy is also Gaussian w/ st

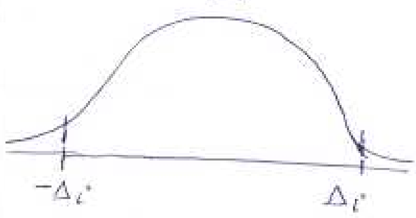
$$\sigma = \sqrt{\sum_{i=1}^{m} c_i \sigma_i^2}$$

Cachy Distribution:

$$\Delta y = \sum c_i \Delta x_i$$

Δx_i are Cauchy w/ param Δ_i then Δy is also cauchy distrib with parameter

$$\Delta = \sum_{i=1}^{m} |c_i| \Delta_i.$$

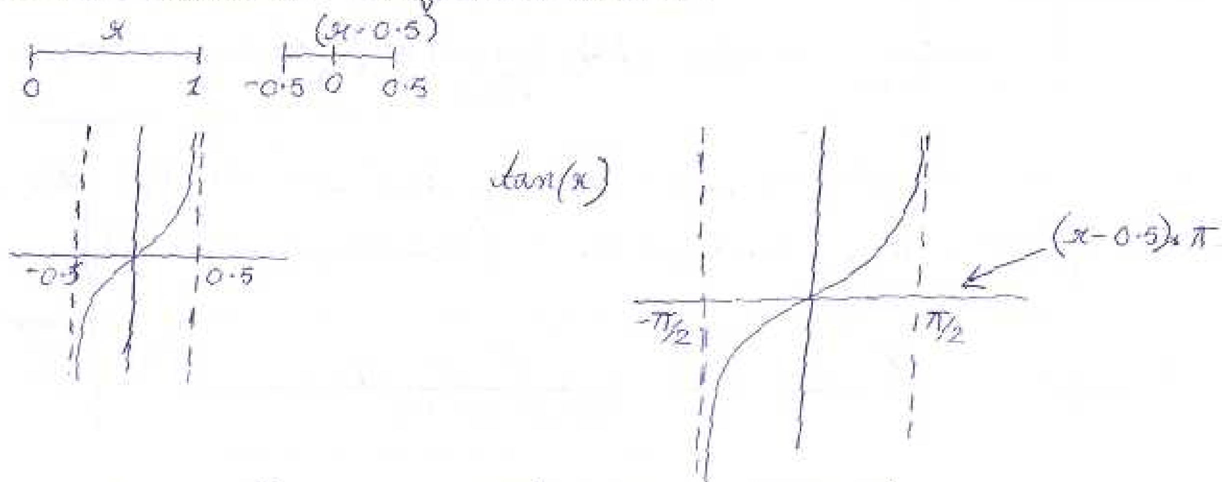


2 questions:

1) how to simulate Cauchy Distribution?

2) how to estimate Δ, based on sample.

How to Simulate Cachy Distribution:



c = tan((rand() - 0.5) * (Math. PI ~~/2.0~~)).

$f(x) \cdot \Delta x$

Cauchy Distribution with delta:

c = Delta * tan((rand() - 0.5) * (Math.PI))