10/21/8

Cauchy distribution

$$\rho(x) = \frac{1}{\pi\Delta}\cdot\frac{1}{1+\frac{x^2}{\Delta}}$$

Let's simplify:

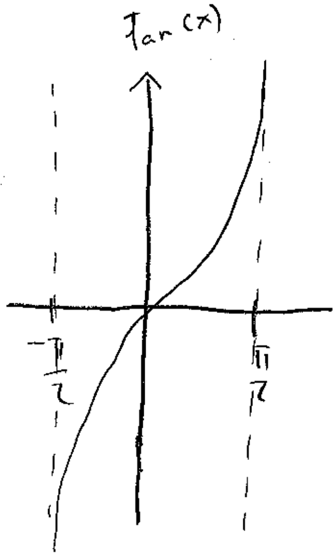
$$y = \frac{x}{\Delta} \qquad \partial y = \frac{\partial x}{\Delta}$$

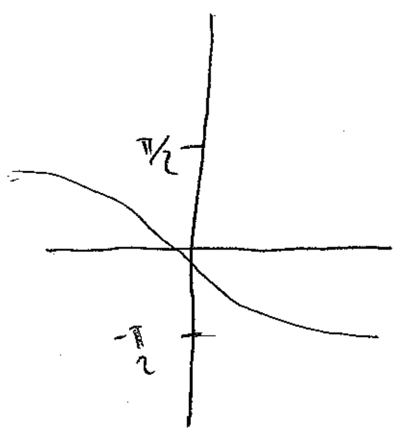
Check:

$$\int_{-\infty}^{\infty} \rho(x)\,\partial x = 1$$

$$\int_{-\infty}^{+\infty} \frac{1}{\pi\Delta}\cdot\frac{\partial x}{1+\frac{x^2}{\Delta}} = \int_{-\infty}^{+\infty} \frac{1}{\pi}\cdot\frac{\partial y}{1+y^2} = \frac{1}{\pi}\cdot\arctan(x)\Big|_{-\infty}^{+\infty}$$

$$= \frac{1}{\pi}\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 1$$





→

$y = \tan(x)$

$x = \arctan(y)$

We want

$$\frac{\partial x}{\partial y} = f(x)$$

$$\frac{\partial y}{\partial x} = \frac{1}{\cos^2 x}, \quad \frac{\partial x}{\partial y} = \frac{1}{\frac{\partial y}{\partial x}} = \boxed{\cos^2 x}$$

$$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\frac{U}{V} = \frac{U'V - UV'}{V^2}$$

$$y' = \frac{\cos x \cdot \cos x - \sin x - \sin x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

We need to describe in terms of y.

$$y^2 = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$$

we can conclude:

$$\frac{1}{\cos^2 x} = y^2 + 1 \qquad \cos^2 x = \frac{1}{y^2+1}$$

logs

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln\left(\frac{1}{b}\right) = \ln 1 - \ln b = -\ln b$$

Max likelyhood Method

we have

$x_1, \ldots, x_n$

Find Δ for which the prob is the largest

$$\frac{1}{\pi\Delta} \cdot \frac{1}{1+\left(\frac{x_1}{\Delta}\right)^2} * \frac{1}{\pi\Delta} \cdot \frac{1}{1+\left(\frac{x_2}{\Delta}\right)^2} * \ldots * \frac{1}{\pi\Delta} \cdot \frac{1}{1+\left(\frac{x_n}{\Delta}\right)^2} \to \underset{\Delta}{\text{Max}}$$

We take log's

$$\frac{1}{\pi^n \Delta^n} = -n\ln\Delta - \ln\left(1+\left(\frac{x_1}{\Delta}\right)^2\right) - \ldots - \ln\left(1+\left(\frac{x_n}{\Delta}\right)^2\right) \to \underset{\Delta}{\text{Max}}$$

Change signs! by min Δ.

$$n\ln\Delta + \ln\left(1+\left(\frac{x_1}{\Delta}\right)^2\right) + \ldots + \ln\left(1+\left(\frac{x_n}{\Delta}\right)^2\right) \to \underset{\Delta}{\text{Min}}$$

$$n\frac{1}{\Delta} + \frac{1}{1+\left(\frac{x_1}{\Delta}\right)^2}\cdot\frac{x_1^2\cdot(-2)}{\Delta^3} + \dots + \frac{1}{1+\left(\frac{x_n}{\Delta}\right)^2}\cdot\frac{x_n^2\cdot(-2)}{\Delta^3} = 0$$

$y = \ln(x)$

$x = e^y$

$\frac{\partial x}{\partial y} = e^y$

$\frac{\partial y}{\partial x} = \frac{1}{e^y} = \frac{1}{x}$

Simplify

$\frac{1}{\Delta}$, move terms,

$$\frac{2x_1^2}{\left(1+\left(\frac{x_1}{\Delta}\right)^2\right)\Delta^2} + \dots + \frac{2x_n^2}{\left(1+\left(\frac{x_n}{\Delta}\right)^2\right)\Delta^2} = n$$

$\boxed{\text{Pg } 29}$

$$\frac{\left(\frac{x_1}{\Delta}\right)^2}{\left(1+\left(\frac{x_1}{\Delta}\right)^2\right)} + \dots + \frac{\left(\frac{x_n}{\Delta}\right)^2}{\left(1+\left(\frac{x_n}{\Delta}\right)^2\right)} = \frac{n}{2}$$

We can simplify more

$$\frac{z}{1+z} = 1 - \frac{1}{1+z}$$

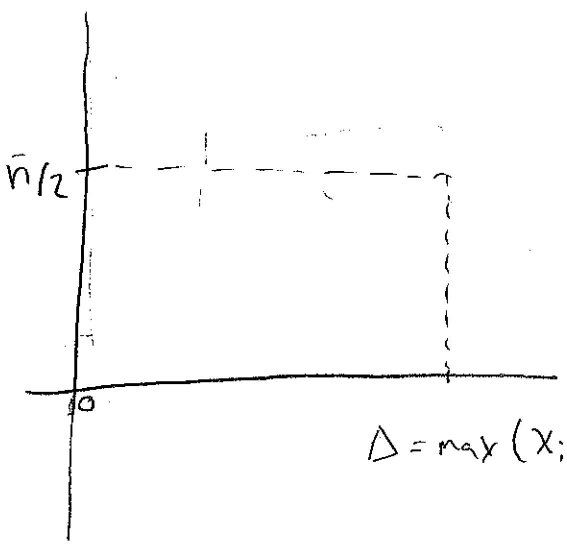
$$1 - \frac{1}{\left(1+\left(\frac{x_n}{n}\right)^2\right)} + \dots + 1 - \frac{1}{\left(1+\left(\frac{x_n}{n}\right)^2\right)} = \frac{n}{2}$$

$\boxed{\text{We have } n\ 1\text{'s}}$

$$n - \frac{1}{\left(1+\left(\frac{x_n}{n}\right)^2\right)} - \dots - \frac{1}{\left(1+\left(\frac{x_n}{n}\right)^n\right)} = \frac{n}{2}$$

$$\frac{1}{1+\left(\frac{x_1}{\Delta}\right)^2}+\ldots+\frac{1}{1+\left(\frac{x_n}{\Delta}\right)^2}=\frac{n}{2}$$

this function is a monotonic fcn of Δ.



We have n terms.
if each is $\geq 1/2$

$1+\left(\frac{x_i}{\Delta}\right)^2 \leq 2$

$\left(\frac{x_i}{\Delta}\right)^2 \leq 1$

$\frac{|x_i|}{\Delta} \leq 1$

$\left(\frac{x_i}{\Delta}\right)^2 \leq 1$

Cauchy-based Monte-Carlo method. (1st approximation)

Given: $\tilde{x}_1, \ldots, \tilde{x}_n$
$\Delta_1, \ldots, \Delta_n$
$f(x_1, \ldots, x_n)$
First we: $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$ $\quad K = 1, \ldots, N$

How: We iterate N times.

* for each i, we simulate Δx_i as Cauchy distr. $\Delta x^{(K)} = \Delta_i \cdot \tan(\pi(\xi - 0.5))$ — random $\in [0,1]$
* $y^{(K)} = f(\tilde{x}_1 - \Delta x_1^{(K)}, \ldots, \tilde{x}_n - \Delta x_n^{(K)})$
* $\Delta y^{(K)} = \tilde{y} - y^{(K)}$

We find Δ from.

$$\sum_{K=1}^{N} \frac{1}{1+\frac{(\Delta y^{(K)})^2}{\Delta^2}} = \frac{N}{2}$$

$\Delta \in [0, \max|\Delta y^{(K)}|]$

binary Search

Correct algorithm

- Compute $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$

  for $K = 1, \ldots, N$

    for each i, we simulate

    $\cancel{\Delta x_i^{(K)}} = C_i^{(K)} \cdot \tan(\pi \cdot (\xi - 0.5))$ // The largest

  * Compute $K = \max |C_i^{(K)}|$

  * $\Delta x_i^{(K)} = \Delta_i \cdot \frac{C_i^{(K)}}{K}$

  * $y^{(K)} = f(\tilde{x}_1 - \Delta x_1^{(K)}, \ldots, \tilde{x}_n - \Delta x_n^{(K)})$

  * $\Delta y^{(K)} = (\tilde{y} - y^{(K)}) \cdot K$

~~then~~

We find Δ from $\sum_{K=1}^{N} \frac{1}{1 + \frac{(\Delta y^{(K)})^2}{\Delta^2}} = \frac{N}{2}$

$\Delta \in [0, \max|\Delta y^{(K)}|]$

```
Lower = 0.0
Upper = ...
while (upper-lower > 0.01) {
    mid = (lower + upper) / 2.0;
    f = F(mid)
    if (f < n/2.0) { lower = mid }
    else { upper = midpoint }
```

```
Sum = 0.0
for (K=1; K<N; K++)
    Sum += 1.0 / (1 + (DeltaY[K] * DeltaY[K]) / (mid * mid));
```

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$$\frac{1}{1+\left(\frac{\Delta y}{\Delta}\right)^2} = \frac{1}{2}$$

(1) One Variable

$$1+\left(\frac{\Delta y}{\Delta}\right)^2 = 2 \qquad \Delta = |\Delta y|$$

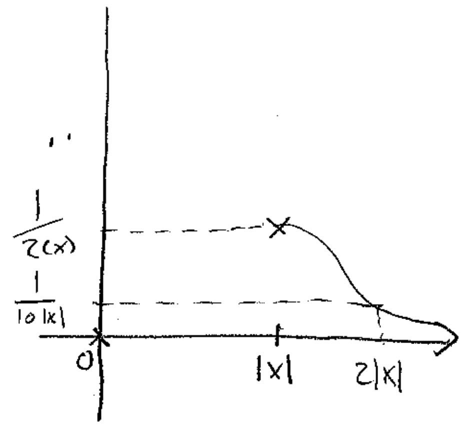
$$\left(\frac{\Delta y}{\Delta}\right)^2 = 1$$

Two Var

$$\frac{1}{\pi\Delta} \cdot \frac{1}{1+\left(\frac{x}{\Delta}\right)^2} = \rho_{Max}(x)$$

$$\Delta = |x| \qquad \Delta = 2|x|$$

$$\frac{1}{2|x|}$$



Please see other notes for this example !!