CS 5354/CS 4365 Advanced Computational Methods in Economics and Finance Fall 2018, Final Exam

Name: _

General comments:

- you are allowed up to 10 pages of handwritten notes;
- if you need extra pages, place your name on each extra page;
- the main goal of most questions is to show that you know the corresponding algorithms; so, if you are running of time, just follow the few first steps of the corresponding algorithm;

Good luck!

1. *Traditional utility-based approach to decision making.*

1a. What is utility? Give a precise definition.

1b. How do you select the very bad alternative A_- and the very good alternative A_+ that are needed to define utility?

1c. Once you have selected the very bad and the very good alternatives, how do we find the utility of a given alternative A? Explain, in detail.

1d. Which of the two actions is better: B or C? For action B, there are two possible outcomes: a_1 with probability 0.3 and utility 100, and a_2 with probability 0.7 and utility −20. For action C, there are also two possible outcomes: b_1 with probability 0.4 and utility 40, and b_2 with probability 0.6 and utility −10.

1a. utility of an event A is the probability of which A has some value as a lottery in which one gets a very good result (A_+) with probability 'u' and a very bad result (A_-) with probability 1-u.

1b. A_+ has to be better than any possible outcome (anything that can encounter)
A_- has to be worse than anything that can be encounter

1c. It is done by bisection; first interval is [0,1], and we find the midpoint (m) by $\frac{\underline{u}+\overline{u}}{2}$, where $\underline{u}=0$ and $\overline{u}=1$ and ask the user to compare alternative A with lottery L(m).

If $A > L(m)$, utility $u(A) \in [m, \overline{u}]$
If $A < L(m)$, utility $u(A) \in [\underline{u}, m]$

The interval chosen is used to continue the process (get new midpoint) until it becomes sufficiently small or it is reached a predefined precision

1d) B a_1 w/prob 0.3 $U=100$
a_2 w/prob 0.7 $U=-20$

C b_1 w/prob 0.4 $U=40$
b_2 w/prob 0.6 $U=-10$

$u(a) = p_1 u(a_1) + p_2 u(a_2)$
$u(a) = (0.3)(100) + (0.7)(-20$
$u(a) = 30 + (-14$
$u(a) = 16$

$u(b) = p_1 u(b_1) + p_2 u(b_2)$
$u(b) = (0.4)(40) + (0.6)(-10)$
$u(b) = 16 + (-6)$
$u(b) = 10$

$\therefore$ action B is better.

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2. *How utility depends on money*

2a. By using the fact that utility is proportional to the square root of money, explain why most people prefer $70 cash to a lottery in which they get $100 with probability 0.7.

2b. Give an example explaining why the simplified assumption -- that utility is proportional to money amount -- does not lead to a reasonable behavior.

2c. What properties were used to justify the power-law dependence of utility on money?

2a. Given that utility is proportional to the square root of money, we have that $u(m)=\sqrt{m}$;

Let $A = u(70) = \sqrt{70} \simeq 8$.

Let B = expected utility of $L(0.7) = 100$

$B = (0.7)(\sqrt{100}) + (0.3)(\sqrt{0})$

$= 7 + 0$

$= 7$

Since $A=8 > B=7$, we can say that this is why most people prefer $70 cash.

2b. Suppose we have two alternatives, $50 in cash & a lottery $100 with probability 0.6.

Assume utility is proportional to money,

$u(m)=m$, $u(50)=50$ and $u(L(0.5)) = 0.6 \times 100 + 0.4 \times 0 = 60$.

People prefer $L(0.6)=\$100$ to $50 cash, which is in reality not true since most would avoid risk and get the $50.

2c. property of scale invariant $u(km)=f(k)\,u(m)$

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3. *Optimal portfolio: Markowitz theory.* Suppose that we have two investments, one with expected return 100 and variance 200, another with expected return 200 and variance 100, and we want to have a return of 140. Assuming that these two investments are independent, find the optimal portfolio.

$\mu_1 = 100$ $\mu_2 = 200$ $\mu = 140$

$\sigma_1^2 = 200$ $\sigma_1^2 = 100$

$$\Sigma_0 = \frac{100^0}{200} + \frac{200^0}{100} = \frac{1}{200} + \frac{1}{100} = \frac{3}{200}$$

$$\Sigma_1 = \frac{100^1}{200} + \frac{200^1}{100} = \frac{5}{2}$$

$$\Sigma_2 = \frac{100^2}{200} + \frac{200^2}{100} = \frac{90000}{200} = 450$$

$$a = \frac{5/2 - 140 \cdot 3/200}{(5/2)^2 - 3/200 \cdot 450} \quad \frac{0.4}{-0.5} = -0.8$$

$$b = \frac{140 \cdot 5/2 - 450}{(5/2)^2 - 3/200 \cdot 450} \quad \frac{-100}{-0.5} = 200$$

$$w_1 = -0.8\left(\frac{100}{200}\right) + \frac{200}{200} = 0.6$$

$$w_2 = -0.8\left(\frac{200}{100}\right) + \frac{200}{100} = 0.4$$

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4. *Decision making under interval uncertainty.* Suppose that:

- for alternative A, the utility is between 5 and 7,
- for alternative B, it is between 4 and 8, and
- for alternative C, it is between 1 and 9.

4a. Which of these three alternatives will be selected:

- by a perfect optimist, for whom the Hurwicz optimism-pessimism parameter is $\alpha_H = 1$?
- by a perfect pessimist, for whom the Hurwicz optimism-pessimism parameter is $\alpha_H = 0$?
- by a realist, for whom the Hurwicz optimism-pessimism parameter is $\alpha_H = 0.5$?

4b. Explain, in detail, how we can deduce the Hurwicz optimism-pessimism formula from the requirement that decision making under interval uncertainty should not change if we change the starting point (shift-invariance) or if we change the measuring unit (scale-invariance).

4a)

Alt	Interval	$\alpha_H = 1$	$\alpha_H = 0$	$\alpha_H = .5$
A	[5, 7]	7	5	$(.5)(5) + (.5)(7) = 6$
B	[4, 8]	8	4	$.5(4) + .5(8) = 6$
C	[1, 9]	9	1	$.5(1) + .5(9) = 5$

Perfect optimist

Perfect pessimist

realist, either A or B

4b) The main idea behind this, is that we need to assign single utility value "u" to each utility interval $[\underline{u}, \overline{u}]$. Since utility is defined modulo linear, re-scaling $u \to k(u+b)$ should not change the assignment. Initially $\alpha_n \in [0,1]$ and utility is defined modulo linear $\to k(\alpha_u + b) = f(\underline{u}, \overline{u})$. Rescaling we have

$[\underline{u} = k(0+b) \to \underline{u} = b.]$

$[\overline{u} = k(1+b) \to \overline{u} = k+b \to k = \overline{u} - \underline{u}]$

Then, $f(\underline{u}, \overline{u}) = f(k(0+b), k(1+b)) \Rightarrow (\overline{u} - \underline{u})\alpha_n + \underline{u} \to$

$\overline{u}\alpha_n - \underline{u}\alpha_n + \underline{u} \Rightarrow \left[f(\underline{u}, \overline{u}) = \overline{u}(\alpha_n) + (1-\alpha)\underline{u}\right]$

Hurwicz Optimism–Pessimism Formula

5. *Probabilistic decision making*

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5a. Suppose that a decision maker follows McFadden's formula with $\beta = \ln(5)$. If we have three alternatives, with utilities 0, 1, and 2, with what probability will the expert select each of these three alternatives?

5b. What property was used to justify McFadden's formula?

5c. If we have three alternatives, with utilities 0, 1, and 2, with what probability will the expert select each of the three alternatives is this expert uses a power-function modification of McFadden's formula: with the first powers? with the third powers?

5d. What property was used to justify the power-function modification of McFadden's formula?

5a. $\beta = \ln(5)$

$a_1 = 0$

$a_2 = 1$

$a_3 = 2$

$$P(a_n) = \frac{\exp(\beta, a(x))}{\sum \exp(\beta u(a))}$$

$$P(a_1) = \frac{5^0}{5^0 + 5^1 + 5^2} = \frac{1}{31}$$

$$P(a_2) = \frac{5^1}{5^0 + 5^1 + 5^2} = \frac{5}{31}$$

$$P(a_3) = \frac{5^2}{5^0 + 5^1 + 5^2} = \frac{25}{31}$$

5b. justified by shift-invariance $F(u+b) = f(b) F(u)$

5c. $u_1 = 0$

$u_2 = 1$

$u_3 = 2$

first power. $P(a_i) = \dfrac{u(a_i)}{\sum u(a_i)}$

$$P(a_1) = \frac{0}{0+1+2} = 0$$

$$P(a_2) = \frac{1}{0+1+2} = \frac{1}{3}$$

$$P(a_3) = \frac{2}{0+1+2} = \frac{2}{3}$$

third power $P(a_i) = \dfrac{u(a_i)^3}{\sum u(a_i)^3}$

$$P(a_1) = \frac{0^3}{0^3+1^3+2^3} = 0$$

$$P(a_2) = \frac{1^3}{0^3+1^3+2^3} = \frac{1}{9}$$

$$P(a_3) = \frac{2^3}{0^3+1^3+2^3} = \frac{8}{9}$$

5d. justified by scale invariance

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6. *Dealing with outliers: robust techniques*. Assume that we have ten estimates for next year's oil prices: three estimate of $60 per barrel, five estimates of $50, and two outliers: an over-pessimistic estimate of $10, and an over-optimistic estimate of $200.

6a. What will be the combined estimate if we use the standard least squares methods (i.e., l^2).

6b. What will be the combined estimate if we use the l^1 method? Explain in what sense this method is more robust.

6c. What is the general class of robust techniques that includes both l^2 and l^1 as particular cases? What is the justification for using methods from this class?

6 -> Data: $n = 10$
[60, 60, 60, 50, 50, 50, 50, 50, 10, 200]

6a) $\frac{3(60) + 5(50) + 10 + 200}{10} = 64$ dollars

6b) [10, 50, 50, 50, 50 | 50, 60, 60, 60, 200]
median
50 dollars

This is more robust because it is not affected by outliers 10 & 200

6c) General class for L^1 & L^2 is L^p method
The estimate $L(e)$ is scale invariant with respect to the error. $L(e) = \text{cost} |e|^p$

7. *Group decision making*

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7a. Suppose that a group of two people needs to select between two alternatives. For the first alternative, their utilities are 5 and 10; for the second alternative, their utilities are 7 and 8. Which alternative should they select if they use Nash's bargaining solution?

7b. What property was used to justify Nash's bargaining solution?

7c. Give an example that Nash's bargaining solution provides a more fair description of the country's economic situation than the Gross Domestic Product (GDP).

7a. $u(a_1) = 5$ $u(b_1) = 7$

$u(a_2) = 10$ $u(b_2) = 8$

$\prod u_A = 5 \times 10 = 50$

$\prod u_B = 7 \times 8 = 56$

since $\prod u_B > \prod u_A$, ∴ alternative B should be selected.

7b. scale invariant is the property to justify the method.

7c. country I country II

$u_1 = 100 = 10^2$ $u_1 = 10$

$u_2 = 1$ $u_2 = 10$

$u_3 = 1$ $u_3 = 10$

$GDP = \sum u_i$

country 1 GDP $= 10^2 + 1 + 1 = 102$

country 2 GDP $= 10 + 10 + 10 = 30$

$NBG = \prod u_i$

country 1 NBG $= 10^2 \times 1 \times 1 = 10^2$

Country 2 NBG $= 10^1 \times 10 \times 10 = 10^3$

∴ solution more fair with NBG.

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8. *How people actually make decisions: peak-end rule.*. An investor placed her money into two hedge funds. The first one led to annual returns of 8%, 1%, 1%, 1%, and 8%. The second one lead to annual returns of 7%, 0%, 7%, 7%, and 7%.

8a. Which of the two investments leads to better end results?

8b. Which of the two investments will the investor prefer if he/she follows the peak-end rule?

8c. What is the justification for the peak-end rule?

1st 8% 1% 1% 1% 8%
2nd 7% 0% 7% 7% 7%

8a. $1.08 \times 1.01 \times 1.01 \times 1.01 \times 1.08 = 1.2017$
$1.07 \times 1 \times 1.07 \times 1.07 \times 1.07 = 1.310$ ← best

8b.

	1st	2nd
1st	8	7
last	8	7
min	1	0
max	8	7
	25	21
	best	

∴ will prefer investment A.

8c. associative, indempotent, scale invariance and shift invariance

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9. *Trade: gravity model*

9a. Let us assume that the trade volume between the two countries is described by the gravity model, with $\alpha = 3$.

- How will the trade volume change if the GDP of the first country increases by 20%?
- How will the trade volume change if the GDP of the second country increases by 30%?
- How will the trade volume be different if countries with the same GDP were located at a twice larger distance than the given pair?

9b. What properties were used to justify the gravity model?

$t = \frac{c g_1 g_2}{r^\alpha}$, $\alpha = 3$ $t = \frac{c g_1 g_2}{r^3}$

9a. 1st country increase 20%, assuming start in 1, 20% = 1.2

$g_1' = 1.2 g_1$
$g_2' = g_2$
$r' = r^\alpha$

$t' = \frac{c(1.2g_1) \times g_2}{r^3} = 1.2\left(\frac{cg_1g_2}{r^3}\right) = 1.2t$

∴ increase of 1.2 change in volume

2nd country increase 30%, assuming start in 1, 30% = 1.3

$g_1' = g_1$
$g_2' = 1.3 g_2$
$r' = r^\alpha$

$t' = \frac{c g_1 \times (1.3 g_2)}{r^3} = \frac{1.3(c g_1 g_2)}{r^3} = 1.3t$

∴ increase of 1.3 change in volume

distance twice larger

$g_1' = g_1$
$g_2' = g_2$
$r' = (2r)^\alpha$

$t' = \frac{c g_1 g_2}{(2r)^3} = \frac{c g_1 g_2}{8r^3} = \frac{1}{8} \times \frac{c g_1 g_2}{r^3} = \frac{1}{8}t$

∴ change of 1/8

9b. scale invariant & additive.

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10. *Possible invariances and resulting formulas*. Provide the detailed proofs of the three main mathematical results that we had:

- that every shift-invariant function F(x), i.e., a function for which F(x + c) = f(c) * F(x) for some f(c), has the form F(x) = const * exp(β * x);
- that every scale-invariant function F(x), i.e., a function for which F(k * x) = f(k) * F(x) for some f(k), has the form F(x) = const * x^α;
- that every additive function, i.e., a function for which F(x + y) = F(x) + F(y), has the form F(x) = const * x.

$F(x+c) = f(c)F(x)$

diff. by c

$F'(c) = f'(c) \times F(x)$

Set $c=0$

$\frac{dF}{dx} = C_0 F$

$C_0 F = f'(\beta)$

separe vars

$\frac{dF}{F} = C_0 \beta dx$

Integrate

$\int \frac{dF}{F} = \int C_0 \beta dx$

$\ln F = C_0 (x)\beta + C_1$

exponent both sides

$F(x) = e^{C_0 (x)\beta + C_1}$

$F(x) = C_1 e^{C_0 (x)\beta}$

where C_1 is a constant.

$F(kx) = f(k)F(x)$

diff. by k

$F'(k) = f'(k)F(x)$

take $k=1$

$\frac{dF}{dx} x = C_\alpha F$

Separe vars

$\frac{dF}{F} = C_\alpha \frac{dx}{x}$

Integrate

$\int \frac{dF}{F} = \int C_\alpha \frac{dx}{x}$

$\ln F = C_\alpha \ln|x| + C_1$

exponent both sides

$f'(x) = e^{C_\alpha \ln(x)} e^{C_1}$

$f'(x) = c(e^{\ln x})^\alpha$

$= C x^\alpha$

where C_1 is a constant

$F(x+y) = F(x) + F(y)$

$\frac{d}{dx} F(x+y) * \frac{d(x+y)}{dy} = \frac{dF(y)}{dy}$

$\frac{dF(x+y)}{dx} = F'(y)$

let $y=0$, $F'(0)=c$

$\frac{d}{dx} F(x) = c$

integrate

$\int dF(x) = \int c\, dx$

$F(x) = cx + c$

where $c=0$

$\therefore F(x) = cx$

where c is a constant

10/10

11. *Least Squares* (For extra credit) Use the Least Squares method to find the values a and b for which $a * x_i + b \sim y_i$, based on the following observations:

- $x_1 = -1$, $y_1 = 1$;
- $x_2 = 0$, $y_2 = 1$;
- $x_3 = 1$, $y_3 = -1$.

$ax + b \simeq y_i$

$\bar{x} = \frac{-1+0+1}{3}$ $\quad \bar{y} = \frac{1+1+(-1)}{3}$ $\quad \overline{xy} = \frac{(-1\times1)+(0\times1)+(1\times-1)}{3}$

$\bar{x} = 0$ $\quad \bar{y} = 1/3$ $\quad \overline{xy} = \frac{-1+0+(-1)}{3} = \frac{-2}{3}$

$\overline{x^2} = \frac{(-1)^2+(0)^2+(1)^2}{3}$

$\overline{x^2} = \frac{1+0+1}{3} = \frac{2}{3}$

$a = \frac{\overline{xy} - \bar{x}\,\bar{y}}{\overline{x^2} - (\bar{x})^2} = \frac{\frac{-2}{3} - (0 \times \frac{1}{3})}{\frac{2}{3} - (0)^2} = \frac{\frac{-2}{3}}{\frac{2}{3}} = -1$ $\quad \underline{a = -1}$ ✓

$b = \bar{y} - a\bar{x}$

$b = \frac{1}{3} - (-1)(0)$

$\underline{b = \frac{1}{3}}$ ✓