Solution to Homework 16

Task. Prove that for the case when $f(a) > f(b)$, linear interpolation is also the only maximally individually robust interpolation.

Solution. When $f(a) > f(b)$, the formula for linear interpolation

$$f_L(x) = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

can be rewritten as

$$f_L(x) = f(a) - \frac{f(a) - f(b)}{b - a} \cdot (x - a).$$

Let us take any value $x \in (a, b)$. Linear interpolation always produces values in between $f(a)$ and $f(b)$, so in this case, $f(a) > f_L(x) > f(b)$.

1. Let us first prove that in this case, we cannot have

$$f(x) < f_L(x) = f(a) - \frac{f(a) - f(b)}{b - a} \cdot (x - a).$$

Indeed, by subtracting, from $f(a)$, both sides of this inequality, we conclude that

$$f(a) - f(x) > f(a) - \left(f(a) - \frac{f(a) - f(b)}{b - a} \cdot (x - a)\right) =$$

$$f(a) - f(a) + \frac{f(a) - f(b)}{b - a} \cdot (x - a) = \frac{f(a) - f(b)}{b - a} \cdot (a - x).$$

Here:

- We have $f(x) < f_L(x)$ and $f_L(x) < f(a)$, so $f(x) < f(a)$. Thus,

$$f(a) - f(x) > 0, \text{ and } |f(a) - f(x)| = f(a) - f(x).$$

- We have $f(a) > f(b)$, so $f(a) - f(b)0$. Thus, $|f(a) - f(b)| = f(a) - f(b)$.
- We have $a < b$, so $b - a > 0$. Thus, $|b - a| = b - a$.
- Also, $a < x$, so $x - a > 0$. Thus, $|x - a| = x - a$.

So, the inequality

$$f(a) - f(x) > \frac{f(a) - f(b)}{b - a} \cdot (x - a)$$

implies that

$$|f(a) - f(x)| > \frac{|f(a) - f(b)|}{|b - a|} \cdot |x - a|.$$

In general, $x' - x'' = -(x'' - x')$, hence $|x' - x''| = |x'' - x'|$. Thus, the above inequality about absolute values can be equivalently rewritten as

$$|f(x) - f(a)| > \frac{|f(b) - f(a)|}{|b - a|} \cdot |x - a|.$$

The ratio $\dfrac{|f(b) - f(a)|}{|b - a|}$ is what we denoted by r. Thus, we get

$$|f(x) - f(a)| > r \cdot |x - a|.$$

However, we assumed that the function $f(x)$ is maximally individually robust and therefore, that $|f(x) - f(x')| \le r \cdot |x - x'|$ for all x and x'. In particular, for $x' = a$, we conclude that

$$|f(x) - f(a)| \le r \cdot |x - a|,$$

which contradicts to the opposite inequality $|f(x) - f(a)| > r \cdot |x - a|$ that we derived. This contradictions shows that we cannot have $f(x) > f_L(x)$.

2. Let us now prove that in this case, we cannot have

$$f(x) > f_L(x) = f(a) - \frac{f(a) - f(b)}{b - a} \cdot (x - a).$$

Indeed, by subtracting $f(b)$ from both sides of this inequality, we conclude that

$$f(x) - f(b) > (f(a) - f(b)) - \frac{f(a) - f(a)}{b - a} \cdot (x - a) =$$

$$(f(a) - f(b)) \cdot \left(1 - \frac{x - a}{b - a}\right) = (f(a) - f(b)) \cdot \frac{b - a - (x - a)}{b - a} =$$

$$(f(a) - f(b)) \cdot \frac{b - x}{b - a} = \frac{f(a) - f(b)}{b - a} \cdot (b - x).$$

Here:

- We have $f(x) > f_L(x)$ and $f_L(x) > f(b)$, so $f(x) > f(b)$. Thus,

  $$f(x) - f(b) > 0, \text{ and } |f(x) - f(b)| = f(x) - f(b).$$

- We have $f(a) > f(b)$, so $f(a) - f(b) > 0$. Thus, $|f(a) - f(b)| = f(a) - f(b)$.

- We have $a < b$, so $b - a > 0$. Thus, $|b - a| = b - a$.
- Also, $x < b$, so $b - x > 0$. Thus, $|b - x| = b - x$.

So, the inequality

$$f(b) - f(x) > \frac{f(b) - f(a)}{b - a} \cdot (b - x)$$

implies that

$$|f(b) - f(x)| > \frac{|f(b) - f(a)|}{|b - a|} \cdot |b - x|.$$

The ratio $\dfrac{|f(b) - f(a)|}{|b - a|}$ is what we denoted by r. Thus, we get

$$|f(b) - f(x)| > r \cdot |b - x|.$$

However, we assumed that the function $f(x)$ is maximally individually robust and therefore, that $|f(x') - f(x'')| \le r \cdot |x' - x''|$ for all x' and x''. In particular, for $x' = b$ and $x'' = x$, we conclude that

$$|f(b) - f(x)| \le r \cdot |b - x|,$$

which contradicts to the opposite inequality $|f(b) - f(x)| > r \cdot |b - x|$ that we derived. This contradictions shows that we cannot have $f(x) < f_L(x)$.

3. Since the value $f(x)$ cannot be larger than $f_L(x)$ and cannot be smaller that $f_L(x)$, it must be exactly equal to $f_L(x)$:

$$f(x) = f_L(x).$$

In other words, the function $f(x)$ must be obtained by linear interpolation.

The statement is proven.