Solution to Problem 19

Task. Let us assume that we know the values $f(a)$ and $f(b)$ for some a and b, and we want to interpolate, i.e., to find the values $f(x)$ for all x between a and b. By definition, the maximally individually robust interpolation $f(x)$ must satisfy the inequality $|f(x)-f(y)| \le r \cdot |x-y|$ for all x and y, where $r = \dfrac{|f(b)-f(a)|}{|b-a|}$. Provide an example of the values x and y showing that when $a=0$, $b=1$, $f(a)=0$, and $f(b)=1$, the function $f(x)=x^2$ is not a maximally individually robust interpolation. *Hint:* it is sufficient to consider values 0, 0.5, and 1.

Solution. In this case, $r = \dfrac{|f(b)-f(a)|}{|b-a|} = \dfrac{1-0}{1-0} = 1$, so maximally individually robust interpolation $f(x)$ must satisfy the inequality

$$|f(x)-f(y)| \le 1 \cdot |x-y| = |x-y|$$

for all x and y.

However, for $x = 0.5$ and $y = 1$, we have

$$|f(x)-f(y)| = |0.5^2 - 1^2| = |0.25 - 1| = 0.75,$$

while $|x-y| = |0.5-a| = 0.5$. So, here, $|f(x)-f(y)| > |x-y|$. Thus, the function $f(x)=x^2$ is not a maximally individually robust interpolation.