Linear Interpolation

Problem.

- We know that the dependence of a quantity y on a quantity x is linear, i.e., that
$$y(x) = a \cdot x + b \tag{1}$$
for some values a and b.

- We know two cases in which we measured x and y:
  - we know the values x_1 and y_1 for which
  $$a \cdot x_1 + b = y_1, \tag{2}$$
  and
  - we know the values x_2 and y_2 for which
  $$a \cdot x_2 + b = y_2. \tag{3}$$

Based on this information, we need to find the formula for $y(x)$.

Solution. Subtracting (2) from (3), we conclude that
$$a \cdot (x_2 - x_1) = y_2 - y_1. \tag{4}$$
Dividing both sides by the difference $x_2 - x_1$, we conclude that
$$a = \frac{y_2 - y_1}{x_2 - x_1}. \tag{5}$$

Subtracting (2) from (1), we conclude that
$$a \cdot (x - x_1) = y - y_1, \tag{6}$$
therefore
$$y = y_1 + a \cdot (x - x_1). \tag{7}$$
Substituting the expression (5) for a into this formula, we conclude that
$$y = y_1 + (x - x_1) \cdot \frac{y_2 - y_1}{x_2 - x_1}. \tag{8}$$
This formula is known as *linear interpolation*.

Example 1. Suppose that we know that:

- for $x_1 = 0$, we have $y_1 = 1$, and
- for $x_2 = 5$, we have $y_2 = 0$.

In this case, we have

$$y = 1 + (x - 0) \cdot \frac{0 - 1}{5 - 0} = 1 + x \cdot \frac{-1}{5} = 1 - \frac{x}{5}. \qquad (9)$$

Example 2. Suppose that we know that:

- for $x_1 = -5$, we have $y_1 = 0$, and
- for $x_2 = 0$, we have $y_2 = 1$.

In this case, we have

$$y = 0 + (x - (-5)) \cdot \frac{1 - 0}{0 - (-5)} = (x + 5) \cdot \frac{1}{5} = \frac{x + 5}{5} = 1 + \frac{x}{5}. \qquad (9)$$