Test 2 for Explainable Fuzzy AI Class Solutions, Summer 2021

Question 1a. What is robustness?

Answer. A function $f(x)$ is called robust if:

- whenever x is close to x' (denoted by $x \approx x'$),
- the values $f(x)$ and $f(x')$ of the function are also close: $f(x) \approx f(x')$.

Question 1b. Why do we want membership functions to be robust?

Answer. In practical applications, the value of the quantity x comes from measurements, and measurements are never absolutely accurate. Anyone who ever measured anything – be it voltage, current, blood pressure, whatever – knows that if we repeat the measurement again, we will get, in general, a slightly different value.

We want to make sure that this difference does not affect the results. For this purpose, we want to make sure that:

- if two measurement results are close, i.e., if $x \approx x'$,
- then the corresponding values of the membership function should also be close: $\mu(x) \approx \mu(x')$.

This is exactly what is called robustness.

Question 1b. Which interpolation is the most robust?

Answer. The most robust is linear interpolation, when, based on the known value $f(a)$ and $f(b)$, we estimate all other values $f(x)$ as

$$f(x) = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a).$$

Question 1c. Why do we want "and"- and "or"-operations to be robust?

Answer. The expert estimate depends on a scale. If we ask the expert to estimate the degree on a scale from 0 to 5, then possible values of the resulting degree are:

$$0/5 = 0.0; \quad 1/5 = 0.2; \quad 2/5 = 0.4; \quad 3/5 = 0.6; \quad 4/5 = 0.8; \text{ and } 5/5 = 1.0.$$

However, if we ask the same expert to estimate his/her degree on a scale from 0 to 4, then we will get different possible values:

$$0/4 = 0.0; \quad 1/4 = 0.25; \quad 2/4 = 0.5; \quad 3/4 = 0.75; \text{ and } 4/4 = 1.0.$$

Suppose that in the first scale, the expert marked 4 on a scale from 0 to 5, leading to an estimate of 0.8. However, no mark on a 0 to 4 scale will lead to the same value 0.8; the closest is the value 0.75 which corresponds to 3 on the 0 to 4 scale. The value 0.75 is close to 0.8, but different.

Similar problem occurs if we use polling: for different numbers of experts, we get different values describing the same degrees of belief. In both cases, the same confidence level of an expert leads, in general, to different degrees $a \neq a'$ – depending on the scale or on the number of experts.

It is therefore reasonable to require that the corresponding small difference $a' - a$ should affect the results as little as possible, i.e., that the "and"- and "or"-operations be robust.

Question 1d. Which "and"- and "or"-operations are the most robust?

Answer. $f_\&(a,b) = a \cdot b$ and $f_\vee(a,b) = a + b - a \cdot b$.

Question 2a. What is variational optimization?

Answer. Variational optimization is an optimization problem in which the unknown is a function.

Question 2b. What is variational derivative and why do we need it?

Answer. Variational derivative is a derivative with respect to a variable which is a value of the unknown function. Why do we need it?

In variational optimization problems, we need to find a function $f(x)$ which is the best – i.e., of which the value of the corresponding criterion $J(f)$ is the smallest (or the largest) possible. Finding a function means funding its value $f(x)$ for all possible inputs x. To find the optimal value $f(x)$, we can differentiate the criterion J with respect to the unknown $f(x)$ – the resulting derivative is what is called variational derivative – and equate this derivative to 0.

Question 2c-d. Suppose that we know that $f(0) = 1$ and $f(1) = 3$, and we want to find the value $f(0.5)$ that minimizes the following expression

$$(f(0) - f(0.5))^2 + (f(0.5) - f(1))^2.$$

Use variational derivative to find this value.

Answer. Differentiating with respect to $f(0.5)$, we get

$$2 \cdot (f(0) - f(0.5)) \cdot (-1) + 2 \cdot (f(0.5) - f(1)) \cdot 1 = 0.$$

Dividing both sides by 2, we get:

$$(f(0) - f(0.5)) \cdot (-1) + (f(0.5) - f(1)) = 0,$$

so

$$f(0.5) - f(0) + f(0.5) - f(1) = 2f(0.5) - f(0) - f(1) = 0.$$

To separate the variable, we add $f(0)$ and $f(1)$ to both sides, getting

$$2f(0.5) = f(0) + f(1),$$

so

$$f(0.5) = \frac{f(0) + f(1)}{2}.$$

In our case, $f(0) = 1$ and $f(1) = 3$, so

$$f(0.5) = \frac{1 + 3}{2} = \frac{4}{2} = 2.$$

Question 3a. Suppose that we know the values $f(a)$ and $f(b)$ for some $a < b$. What does it mean for an interpolating function $f(x)$ to be individually robust? Provide a precise definition.

Answer. Individually robust means that for all x and x', the function $f(x)$ satisfies the inequality $|f(x) - f(x')| \le K \cdot |x - x'|$ with the smallest possible value K.

Question 3b. Which interpolation is the most individually robust?

Answer. The most individually robust is linear interpolation.

Question 3c. Provide an example showing that when we know that $f(0) = 1$ and $f(1) = 0$, the function $f(x) = (1-x)^2$ is not maximally individually robust.

Answer. For the function $f_L(x) = 1 - x$, we have $|f_L(x) - f_L(x')| \le |x - x'|$, so the desired inequality is satisfied for $K = 1$. However, for the function $f(x) = (1-x)^2$, we have $f(0) = (1-0)^2 = 1^2 = 1$ and $f(0.5) = (1-0.5)^2 = 0.5^2 = 0.25$. So, for $x = 0$ and $x' = 0.5$, we get

$$|f(x) - f(x')| = |f(0) - f(0.5)| = |1 - 0.25| = 0.75$$

and $|x - x'| = |0 - 0.5| = 0.5$, so

$$|f(x) - f(x')| > |x - x'|,$$

while in the maximally individually robust case, we should have

$$|f(x) - f(x')| \le |x - x'|.$$

Question 3d. Which "and"-operation is the most individually robust?

Answer. $f_\&(a, b) = \min(a, b)$.

Question 3e. Provide an example showing that algebraic product is not maximally individually robust.

Answer. For min, we have $|\min(a, b) - \min(a', b')| \le \max(|a - a'|, |b - b'|)$, i.e., we have individual robustness with $K = 1$. However, for the algebraic product $f_\&(a, b) = a \cdot b$, for $a = b = 0.5$ and $a' = b' = 1$, we have

$$|f_\&(a, b) - f_\&(a', b')| = |a \cdot b - a' \cdot b'| = |0.5 \cdot 0.5 - 1 \cdot 1| = |0.25 - 1| = 0.75,$$

while

$$\max(|a - a'|, |b - b'|) = \max(|0.5 - 1|, |0.5 - 1|) = \max(0.5, 0.5) = 0.5.$$

So here,

$$|f_\&(a, b) - f_\&(a', b')| = 0.75 > \max(|a - a'|, |b - b'|) = 0.5.$$

Thus, algebraic product is not maximally individually robust, because that would mean, in particular, that

$$|f_\&(a, b) - f_\&(a', b')| \le \max(|a - a'|, |b - b'|).$$

Question 4.

- Suppose that we know that $f(0) = 0$, $f(1) = 1$, and $f(2) = 4$.
- Use the least squares formulas to come up with the best linear approximation to this data.

Answer. Here, $K = 3$,

$$\overline{x} = x^{(1)} + x^{(2)} + x^{(3)} = 0 + 1 + 2 = 3,$$

$$\overline{y} = y^{(1)} + y^{(2)} + y^{(3)} = 0 + 1 + 4 = 5;$$

$$\overline{x^2} = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2 + \left(x^{(3)}\right)^2 = 0^2 + 1^2 + 2^2 = 0 + 1 + 4 = 5;$$

$$\overline{x \cdot y} = x^{(1)} \cdot y^{(1)} + x^{(2)} \cdot y^{(2)} + x^{(3)} \cdot y^{(3)} =$$

$$0 \cdot 0 + 1 \cdot 1 + 2 \cdot 4 = 0 + 1 + 8 = 9.$$

So,

$$c_1 = \frac{K \cdot \overline{x \cdot y} - \overline{x} \cdot \overline{y}}{K \cdot \overline{x^2} - (\overline{x})^2} = \frac{3 \cdot 9 - 3 \cdot 5}{3 \cdot 5 - 3^2} = \frac{27 - 15}{15 - 9} = \frac{12}{6} = 2$$

and

$$c_2 = \frac{\overline{x^2} \cdot \overline{y} - \overline{x} \cdot \overline{x \cdot y}}{K \cdot \overline{x^2} - (\overline{x})^2} = \frac{5 \cdot 5 - 3 \cdot 9}{3 \cdot 5 - 3^2} = \frac{25 - 27}{6} = \frac{-2}{6} = -\frac{1}{3} = -0.33\ldots$$

Question 5a-b. Which "and" and "or"-operations should we use in the following two situations:

- if we are controlling a group of objects, and malfunctioning of one of them is OK as long as, on average, they all fulfil their mission;
- if we are controlling a single object.

Answer.

- If we are controlling a group of objects, and we want to achieve the best overall result, we should use

$$f_{\&}(a,b) = a \cdot b \text{ and } f_{\vee}(a,b) = a + b - a \cdot b.$$

- If we are controlling an individual object, and we want to achieve the best result for this object, we should use

$$f_{\&}(a,b) = \min(a,b) \text{ and } f_{\vee}(a,b) = \max(a,b).$$

Question 5c. So, how can we use fuzzy techniques in explainable AI?

Answer. We start with expert rules – this what makes this approach explainable. We then use general fuzzy methodology – explained in the previous lectures – to find the first-approximation dependence $y = f(x_1, \ldots, x_n)$.

When applying the fuzzy methodology, we used some parameters – e.g., for negligible, we selected 5 as the borderline value starting with which the difference is absolutely not negligible. The choice of these parameters is rather arbitrary. For example, to describe what is negligible, we could use 4 or 6 instead of 5.

So, instead of picking a single such value:

- we make this value a parameter, and then
- we find the values of all these parameters for which, for each k, the predictions of the resulting fuzzy system are the closest to the desired value $y^{(k)}$.

Question 5d. What is tuning and how is it different from machine learning?

Answer. We have:

- the data $\left(x^{(k)}, y^{(k)}\right)$, and
- an algorithm $f_0(x)$ that fits the data, e.g., for which $f_0\left(x^{(k)}\right) \approx y^{(k)}$.

Tuning means finding an algorithm $f(x)$ that provides a better fit with the data.

The difference from machine learning is that, in addition to the data, we also have an algorithm $f_0(x)$ that fits the data.