Detailed Proof of the Result About Scale-Invariant Functions

Definition. *A function* $f(x)$ *is called* scale-invariant *if for every* $\lambda > 0$*, there exists a* $\mu > 0$ *such that* $y = f(x)$ *implies* $y' = f(x')$*, where we denoted* $y' = \mu \cdot y$ *and* $x' = \lambda \cdot x$*.*

Proposition 1. *For every two real numbers* A *and* a*, the function* $y = A \cdot x^a$ *is scale-invariant.*

Comment. In this and other proofs, we will add reminders about needed facts from algebra and calculus.

Proof. Let us assume that $y = f(x)$, i.e., for this function $f(x)$, that

$$y = A \cdot x^a.$$

What can we then say about

$$y' = f(x') = A \cdot (x')^a?$$

Substituting $x' = \lambda \cdot x$ into this expression, we get

$$y' = A \cdot (\lambda \cdot x)^a.$$

Reminder. It known, from algebra, that $(A \cdot B)^c = A^c \cdot B^c$*.*

Thus, we get

$$y' = A \cdot \lambda^a \cdot x^a.$$

Since multiplication is commutative, we get

$$y' = \lambda^a \cdot A \cdot x^a = \lambda^a \cdot (A \cdot x^a).$$

Here,

$$A \cdot x^a = y,$$

so we conclude that

$$y' = \lambda^a \cdot y.$$

If we denote λ^a by μ, we get

$$y' = \mu \cdot y.$$

So, for $\mu = \lambda^a$, we have proved exactly what we wanted: that if $y = f(x)$, then $y' = f(x')$ for $y' = \mu \cdot y$. In this case, $\mu = \lambda^a$. The proposition is proven.

Comment. We have proven that all functions of the type $f(x) = A \cdot x^a$ are scale-invariant. Let us now prove that only such functions are scale-invariant.

Proposition 2. *If a differentiable function $f(x)$ is scale-invariant, then it is equal to $f(x) = A \cdot x^a$ for some A and a.*

Proof. Let us assume that the differentiable function $f(x)$ is scale-invariant. By definition of scale-invariance, this means that for every $\lambda > 0$, there exists some value μ (depending on λ) for which $y = f(x)$ implies that $y' = f(x')$, where $y' = \mu \cdot y$ and $x' = \lambda \cdot x$. Since μ depends in λ, let us write this dependence in explicit form $\mu = \mu(\lambda)$.

Let us take any x and take $y = f(x)$. Then, for each λ, we have $y' = f(x')$, where $y' = \mu(\lambda) \cdot y$ and $x' = \lambda \cdot x$. Substituting these expressions for y' and x' into the formula $y' = f(x')$, we conclude that

$$\mu(\lambda) \cdot y = f(\lambda \cdot x).$$

Here, by our choice of y, we have $y = f(x)$. Substituting $f(x)$ instead of y into the above equality, we get

$$\mu(\lambda) \cdot f(x) = f(\lambda \cdot x).$$

Let us now differentiate both sides of this equality with respect to λ: since the functions are equal, their derivatives should be equal too.

With respect to λ, the term $f(x)$ – that does not depend on λ – is a constant. Thus, the derivative of the leftohand side takes the form

$$\frac{d\mu}{d\lambda} \cdot f(x).$$

Reminder. We used the fact that

$$\frac{d}{dx}(C \cdot f(x)) = C \cdot \frac{df}{dx}.$$

To compute the derivative of the right-hand side, we use the chain rule, and get

$$\frac{d}{d\lambda} f(\lambda \cdot x) = \frac{df}{dx}(\lambda \cdot x) \cdot \frac{d}{d\lambda}(\lambda \cdot x) = \frac{df}{dx}(\lambda \cdot x) \cdot x.$$

Reminder. Chain rule is $(f(g(x))' = f'(g(x)) \cdot d'(x)$. We also used the fact that the derivative of the linear function is equal to the coefficient at the variable, in particular:

$$\frac{d}{d\lambda}(\lambda \cdot x) = x.$$

Thus, the equality between derivatives of the left-hand side and of the right-hand side takes the form

$$\frac{d\mu}{d\lambda} \cdot f(x) = \frac{df}{dx}(\lambda \cdot x) \cdot x.$$

This equality is true for every $\lambda > 0$. To simplify this equality, let us take $\lambda = 1$, and let us denote by a the value of the derivative $\frac{d\mu}{d\lambda}$ for $\lambda = 1$. Then, we get the following:

$$a \cdot f = \frac{df}{dx} \cdot x.$$

Let us separate the variables. For this, we divide both sides of this equality by x and by f and multiply both sides by dx. Then, we get:

$$a \cdot \frac{dx}{x} = \frac{df}{f}.$$

Now, we integrate both sides, and get

$$a \cdot \ln(x) + C = \ln(f),$$

where C is the integration constant.

Reminder. We used the fact that

$$\int \frac{1}{x}\, dx = \ln(x) + C.$$

Now we have an expression for the logarithm $\ln(f)$ of the desired function $Ef(x)$. To get the expression for the desired function $f(x)$ itself, let us apply $\exp(z) = e^a$ to both sides. Then, we get

$$e^{a \cdot \ln(x) + C} = e^{\ln(f)} = f.$$

Reminder. Here, we used the definition of the logarithm: $\log_a(x)$ *is the lower to which we need to raise a to get x, i.e.,f or which* $a^{\log_a(x)} = x$. *Natural logarithm is simply logarithm base* $e = 2.78\ldots$: $\ln(x) = \log_e(x)$. *Thus, by definition of the logarithm,* $e^{\log_e(f)} = f$, *i.e., indeed,* $e^{\ln(f)} = f$.

The left0hand side is equal to $e^{a \cdot \ln(f)} \cdot e^C$. So, if we denote e^C by A, we get

$$f(x) = A \cdot e^{a \cdot \ln(x)}.$$

Reminder. We used the fact from algebra that, in general, $A^{b+c} = A^b \cdot A^c$. *This is easy to remember:*

- A^b *means* $A \cdot A \cdot \ldots \cdot A$, *where the multiplication is repeated* b *times, and*
- A^c *means* $A \cdot A \cdot \ldots \cdot A$, *where the multiplication is repeated* c *times.*

Thus,

$$A^b \cdot A^c = A \cdot A \cdot \ldots \cdot A \text{ (}b\text{ times)} \cdot A \cdot A \cdot \ldots \cdot A \text{ (}c\text{ times)},$$

overall A *is multiplied by itself* $b + c$ *times, so indeed* $A^{b+c} = A^b \cdot A^c$.

In the following step, we will another fact from algebra, that $(A^b)^c = A^{b \cdot c}$. *Indeed,* A^b *means* $A \cdot A \cdot \ldots \cdot A$, *where the multiplication is repeated* b *times. Thus,* $(A^b)^c$ *means that we have* A^b *repeated* c *times:*

$$(A^b)^c = A^b \cdot \ldots \cdot A^b \text{ (}c\text{ times)}.$$

Substituting the expression for A^b *into this formula, we get:*

$$(A^b)^c = A \cdot A \cdot \ldots \cdot A \text{ (}b\text{ times)} \cdot \ldots \cdot A \cdot A \cdot \ldots \cdot A \text{ (}b\text{ times)}.$$

We have c *groups each of which has* b *A's. Thus, overall, we have* $b \cdot c$ *A's, i.e., indeed,* $(A^b)^c = A^{b \cdot c}$.

Here, $e^{a \cdot \ln(x)} = \left(e^{\ln(x)}\right)^a$. We already know that $e^{\ln(x)} = x$, so $e^{a \cdot \ln(x)} = x^a$, and the above formula gets the desired form

$$y = A \cdot x^a.$$

The proposition is proven.