Solution to Problem 10

Task. Show that the class of all functions obtained from

$$y = \frac{1}{1 + \exp(-k \cdot x)}$$

by fractional-linear transformations is shift-invariant.

Solution. If we apply shift $x \mapsto x + x_0$ to the above expression, we get

$$Y = \frac{1}{1 + \exp(-k \cdot (x + x_0))}.$$

Let us show that Y can be obtained from y by a fractional-linear transformation. To get the expression for Y in terms of y, we can do the following:

- use the known expression for $y(x)$ to express x in terms of y, and
- substitute the resulting expression $x(y)$ into the above formula $Y = Y(x)$; this way, we get $Y = Y(x(y))$, i.e., the desired expression for Y in terms of y.

To get the expression for x in terms of y, let is try to move all the terms related to x to one side of the equality and all the other terms to the other side. First, we can somewhat simplify the relation between x and y if we take inverses of both sides, then we get

$$\frac{1}{y} = 1 + \exp(-k \cdot x).$$

To separate the variables, we subtract 1 from both sides, resulting in

$$\exp(-k \cdot x) = \frac{1}{y} - 1.$$

The value x is under the exponential function, so to get it back we need to apply an inverse function – which is the logarithm. By applying natural logarithm to both sides, we get

$$-k \cdot x = \ln\left(\frac{1}{y} - 1\right),$$

so

$$x = \frac{1}{-k} \cdot \ln\left(\frac{1}{y} - 1\right).$$

Let us now substitute this expression for $x(y)$ into the formula for $Y(x)$. To compute Y, we:

- first compute $x + x_0$,
- then we compute $-k \cdot (x + x_0)$,
- then, we compute $\exp(-k \cdot (x + x_0))$,
- then, we add 1 to this value, and
- finally, we divide 1 by the resulting sum.

Let us show, step by step, how this computation will look like if we substitute the above $x(y)$ instead of x.

- First, we compute

$$x + x_0 = \frac{1}{-k} \cdot \ln\left(\frac{1}{y} - 1\right) + x_0.$$

- Then, we compute

$$-k \cdot (x + x_0) = -k \cdot x - k \cdot x_0 = -k \cdot \frac{1}{-k} \cdot \ln\left(\frac{1}{y} - 1\right) - k \cdot x_0.$$

  In the first term, we first divide by $-k$, then multiply by $-k$, so this term can be simplified:

$$-k \cdot (x + x_0) = \ln\left(\frac{1}{y} - 1\right) - k \cdot x_0.$$

- Then, we compute

$$\exp(-k \cdot (x + x_0)) = \exp\left(\ln\left(\frac{1}{y} - 1\right) - k \cdot x_0\right).$$

  It is known that $\exp(a + b) = \exp(a) \cdot \exp(b)$, so

$$\exp(-k \cdot (x + x_0)) = \exp\left(\ln\left(\frac{1}{y} - 1\right)\right) \cdot \exp(-k \cdot x_0).$$

  By definition, natural logarithm $\ln(a)$ is the power to which we need to raise e to get a, so $\exp(\ln(a)) = a$. Thus, the above expression takes the form

$$\exp(-k \cdot (x + x_0)) = \left(\frac{1}{y} - 1\right) \cdot \exp(-k \cdot x_0).$$

  To simplify this expression, we can add the two fractions in the right-hand side, and get

$$\exp(-k \cdot (x + x_0)) = \frac{1 - y}{y} \cdot \exp(-k \cdot x_0) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0)}{y}.$$

- Now, we can add 1 to this value, and get

$$1 + \exp(-k \cdot (x + x_0) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0)}{y} + 1.$$

  If we add the two fractions in the right-hand side, we get

$$1 + \exp(-k \cdot (x + x_0) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0) + y}{y} =$$

$$\frac{(1 - \exp(-k \cdot x_0)) \cdot y + \exp(-k \cdot x_0)}{y}.$$

- Finally, we divide 1 by the resulting fraction, which means we swap the numerator and the denominator:

$$Y = \frac{y}{(1 - \exp(-k \cdot x_0)) \cdot y + \exp(-k \cdot x_0)}.$$

We indeed get the expression for Y as a ratio of two linear functions of y. So, Y can indeed obtained from y by a fractional-linear transformation.

Comment. This derivation can be simplified if we take into account that

$$\exp(-k \cdot (x + x_0)) = \exp(-k \cdot x - k \cdot x_0) = \exp(-k \cdot x) \cdot \exp(-k \cdot x_0).$$

We know that

$$\exp(-k \cdot x) = \frac{1}{y} - 1,$$

so

$$\exp(-k \cdot (x + x_0)) = \left(\frac{1}{y} - 1\right) \cdot \exp(-k \cdot x_0).$$

By adding the fractions in the right-hand side, we get:

$$\exp(-k \cdot (x + x_0)) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0)}{y}.$$

Now, we can add 1 to this value, and get

$$1 + \exp(-k \cdot (x + x_0) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0)}{y} + 1.$$

If we add the two fractions in the right-hand side, we get

$$1 + \exp(-k \cdot (x + x_0) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0) + y}{y} =$$

$$\frac{(1 - \exp(-k \cdot x_0)) \cdot y + \exp(-k \cdot x_0)}{y}.$$

Finally, we divide 1 by the resulting fraction, which means we swap the numerator and the denominator:

$$Y = \frac{y}{(1 - \exp(-k \cdot x_0)) \cdot y + \exp(-k \cdot x_0)}.$$