Solution to Problem 10

Task. Show that the class of all functions obtained from

\[ y = \frac{1}{1 + \exp(-k \cdot x)} \]

by fractional-linear transformations is shift-invariant.

Solution. If we apply shift \( x \mapsto x + x_0 \) to the above expression, we get

\[ Y = \frac{1}{1 + \exp(-k \cdot (x + x_0))}. \]

Let us show that \( Y \) can be obtained from \( y \) by a fractional-linear transformation. To get the expression for \( Y \) in terms of \( y \), we can do the following:

- use the known expression for \( y(x) \) to express \( x \) in terms of \( y \), and
- substitute the resulting expression \( x(y) \) into the above formula \( Y = Y(x) \); this way, we get \( Y = Y(x(y)) \), i.e., the desired expression for \( Y \) in terms of \( y \).

To get the expression for \( x \) in terms of \( y \), let is try to move all the terms related to \( x \) to one side of the equality and all the other terms to the other side. First, we can somewhat simplify the relation between \( x \) and \( y \) if we take inverses of both sides, then we get

\[ \frac{1}{y} = 1 + \exp(-k \cdot x). \]

To separate the variables, we subtract 1 from both sides, resulting in

\[ \exp(-k \cdot x) = \frac{1}{y} - 1. \]

The value \( x \) is under the exponential function, so to get it back we need to apply an inverse function – which is the logarithm. By applying natural logarithm to both sides, we get

\[ -k \cdot x = \ln \left( \frac{1}{y} - 1 \right), \]

so

\[ x = \frac{1}{-k} \cdot \ln \left( \frac{1}{y} - 1 \right). \]

Let us now substitute this expression for \( x(y) \) into the formula for \( Y(x) \). To compute \( Y \), we:
• first compute \( x + x_0 \).
• then we compute \(-k \cdot (x + x_0)\),
• then, we compute \(\exp(-k \cdot (x + x_0))\),
• then, we add 1 to this value, and
• finally, we divide 1 by the resulting sum.

Let us show, step by step, how this computation will look like if we substitute the above \( x(y) \) instead of \( x \).

• First, we compute

\[
x + x_0 = \frac{1}{-k} \cdot \ln \left( \frac{1}{y} - 1 \right) + x_0.
\]

• Then, we compute

\[
-k \cdot (x + x_0) = -k \cdot x - k \cdot x_0 = -k \cdot \frac{1}{-k} \cdot \ln \left( \frac{1}{y} - 1 \right) - k \cdot x_0.
\]

In the first term, we first divide by \(-k\), then multiply by \(-k\), so this term can be simplified:

\[
-k \cdot (x + x_0) = \ln \left( \frac{1}{y} - 1 \right) - k \cdot x_0.
\]

• Then, we compute

\[
\exp(-k \cdot (x + x_0)) = \exp \left( \ln \left( \frac{1}{y} - 1 \right) - k \cdot x_0 \right).
\]

It is known that \(\exp(a + b) = \exp(a) \cdot \exp(b)\), so

\[
\exp(-k \cdot (x + x_0)) = \exp \left( \ln \left( \frac{1}{y} - 1 \right) \right) \cdot \exp(-k \cdot x_0).
\]

By definition, natural logarithm \(\ln(a)\) is the power to which we need to raise \(e\) to get \(a\), so \(\exp(\ln(a)) = a\). Thus, the above expression takes the form

\[
\exp(-k \cdot (x + x_0)) = \left( \frac{1}{y} - 1 \right) \cdot \exp(-k \cdot x_0).
\]

To simplify this expression, we can add the two fractions in the right-hand side, and get

\[
\exp(-k \cdot (x + x_0)) = \frac{1 - y}{y} \cdot \exp(-k \cdot x_0) = -\frac{\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0)}{y}.
\]
• Now, we can add 1 to this value, and get
\[ 1 + \exp(-k \cdot (x + x_0)) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0)}{y} + 1. \]
If we add the two fractions in the right-hand side, we get
\[ 1 + \exp(-k \cdot (x + x_0)) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0) + y}{y} = \frac{(1 - \exp(-k \cdot x_0)) \cdot y + \exp(-k \cdot x_0)}{y}. \]
• Finally, we divide 1 by the resulting fraction, which means we swap the numerator and the denominator:
\[ Y = \frac{y}{(1 - \exp(-k \cdot x_0)) \cdot y + \exp(-k \cdot x_0)}. \]

We indeed get the expression for \( Y \) as a ratio of two linear functions of \( y \). So, \( Y \) can indeed obtained from \( y \) by a fractional-linear transformation.

Comment. This derivation can be simplified if we take into account that
\[ \exp(-k \cdot (x + x_0)) = \exp(-k \cdot x - k \cdot x_0) = \exp(-k \cdot x) \cdot \exp(-k \cdot x_0). \]
We know that \( \exp(-k \cdot x) = \frac{1}{y} - 1 \), so
\[ \exp(-k \cdot (x + x_0)) = \left( \frac{1}{y} - 1 \right) \cdot \exp(-k \cdot x_0). \]

By adding the fractions in the right-hand side, we get:
\[ \exp(-k \cdot (x + x_0)) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0)}{y}. \]
Now, we can add 1 to this value, and get
\[ 1 + \exp(-k \cdot (x + x_0)) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0)}{y} + 1. \]
If we add the two fractions in the right-hand side, we get
\[ 1 + \exp(-k \cdot (x + x_0)) = \frac{-\exp(-k \cdot x_0) \cdot y + \exp(-k \cdot x_0) + y}{y} = \frac{(1 - \exp(-k \cdot x_0)) \cdot y + \exp(-k \cdot x_0)}{y}. \]
Finally, we divide 1 by the resulting fraction, which means we swap the numerator and the denominator:
\[ Y = \frac{y}{(1 - \exp(-k \cdot x_0)) \cdot y + \exp(-k \cdot x_0)}. \]