Solution to Problem 11

Task. Show that for each a, the class of all functions obtained from $1/(1+x^a)$ by fractional-linear transformations is scale-invariant.

Solution. If we apply scaling $x \mapsto \lambda \cdot x$, i.e., replace x with $\lambda \cdot x$ in the above expression, we get

$$Y = \frac{1}{1 + (\lambda \cdot x)^a}.$$

Since $(A \cdot B)^a = A^a \cdot B^a$, we get

$$Y = \frac{1}{1 + \lambda^a \cdot x^a}.$$

Let us show that Y can be obtained from y by a fractional-linear transformation. To get the expression for Y in terms of y, we can do the following:

- use the known expression for $y(x)$ to express x in terms of y, and
- substitute the resulting expression $x(y)$ into the above formula $Y = Y(x)$; this way, we get $Y = Y(x(y))$, i.e., the desired expression for Y in terms of y.

To get the expression for x in terms of y, let is try to move all the terms related to x to one side of the equality and all the other terms to the other side. First, we can somewhat simplify the relation between x and y if we take inverses of both sides, then we get

$$\frac{1}{y} = 1 + x^a.$$

To separate the variables, we subtract 1 from both sides, resulting in

$$x^a = \frac{1}{y} - 1.$$

To simplify this expression, we can add the two fractions in the right-hand side, and get

$$x^a = \frac{1 - y}{y}.$$

To get the expression for x, we need to apply an inverse function – which is raising to the power $1/a$. By applying this inverse function to both sides, we get

$$x = \left(\frac{1 - y}{y}\right)^{1/a}.$$

Let us now substitute this expression for $x(y)$ into the formula for $Y(x)$. To compute Y, we:

- first, we compute x^a,
- then, we compute $\lambda^a \cdot x^a$,
- after that, we add 1 to this value, and
- finally, we divide 1 by the resulting sum.

Let us show, step by step, how this computation will look like if we substitute the above $x(y)$ instead of x.

- First, we compute
$$x^a = \left(\left(\frac{1-y}{y}\right)^{1/a}\right)^a = \frac{1-y}{y}.$$
- Then, we compute
$$\lambda^a \cdot x^a = \lambda^a \cdot \frac{1-y}{y} = \frac{\lambda^a - \lambda^a \cdot y^a}{y}.$$
- After that, we compute
$$1 + \lambda^a \cdot x^a = 1 + \frac{\lambda^a - \lambda^a \cdot y^a}{y}.$$
To simplify this expression, we can add the two fractions in the right-hand side, and get
$$1 + \lambda^a \cdot x^a = \frac{y + \lambda^a - \lambda^a \cdot y^a}{y} = \frac{\lambda^a + (1-\lambda^a) \cdot y}{y}.$$
- Finally, we compute
$$Y = \frac{y}{\lambda^a + (1-\lambda^a) \cdot y}.$$

We indeed get the expression for Y as a ratio of two linear functions of y. So, Y can indeed obtained from y by a fractional-linear transformation.

Comment. This derivation can be simplified if we take into account that there is no need to first find transform x^a into x and then again compute x^a. So, first, we can somewhat simplify the relation between x and y if we take inverses of both sides, then we get
$$\frac{1}{y} = 1 + x^a.$$
To separate the variables, we subtract 1 from both sides, resulting in
$$x^a = \frac{1}{y} - 1.$$

To simplify this expression, we can add the two fractions in the right-hand side, and get

$$x^a = \frac{1-y}{y}.$$

Then, we compute

$$\lambda^a \cdot x^a = \lambda^a \cdot \frac{1-y}{y} = \frac{\lambda^a - \lambda^a \cdot y^a}{y}.$$

After that, we compute

$$1 + \lambda^a \cdot x^a = 1 + \frac{\lambda^a - \lambda^a \cdot y^a}{y}.$$

To simplify this expression, we can add the two fractions in the right-hand side, and get

$$1 + \lambda^a \cdot x^a = \frac{y + \lambda^a - \lambda^a \cdot y^a}{y} = \frac{\lambda^a + (1 - \lambda^a) \cdot y}{y}.$$

Finally, we compute

$$Y = \frac{y}{\lambda^a + (1 - \lambda^a) \cdot y}.$$