Solution to Problem 11

**Task.** Show that for each $a$, the class of all functions obtained from $1/(1 + x^a)$ by fractional-linear transformations is scale-invariant.

**Solution.** If we apply scaling $x \mapsto \lambda \cdot x$, i.e., replace $x$ with $\lambda \cdot x$ in the above expression, we get

$$Y = \frac{1}{1 + (\lambda \cdot x)^a}.$$

Since $(A \cdot B)^a = A^a \cdot B^a$, we get

$$Y = \frac{1}{1 + \lambda^a \cdot x^a}.$$

Let us show that $Y$ can be obtained from $y$ by a fractional-linear transformation. To get the expression for $Y$ in terms of $y$, we can do the following:

- use the known expression for $y(x)$ to express $x$ in terms of $y$, and
- substitute the resulting expression $x(y)$ into the above formula $Y = Y(x)$; this way, we get $Y = Y(x(y))$, i.e., the desired expression for $Y$ in terms of $y$.

To get the expression for $x$ in terms of $y$, let’s try to move all the terms related to $x$ to one side of the equality and all the other terms to the other side. First, we can somewhat simplify the relation between $x$ and $y$ if we take inverses of both sides, then we get

$$\frac{1}{y} = 1 + x^a.$$

To separate the variables, we subtract 1 from both sides, resulting in

$$x^a = \frac{1}{y} - 1.$$

To simplify this expression, we can add the two fractions in the right-hand side, and get

$$x^a = \frac{1 - y}{y}.$$

To get the expression for $x$, we need to apply an inverse function – which is raising to the power $1/a$. By applying this inverse function to both sides, we get

$$x = \left(\frac{1 - y}{y}\right)^{1/a}.$$
Let us now substitute this expression for $x(y)$ into the formula for $Y(x)$. To compute $Y$, we:

- first, we compute $x^a$,
- then, we compute $\lambda^a \cdot x^a$,
- after that, we add 1 to this value, and
- finally, we divide 1 by the resulting sum.

Let us show, step by step, how this computation will look like if we substitute the above $x(y)$ instead of $x$.

- First, we compute
  \[
x^a = \left( \left( \frac{1 - y}{y} \right)^{1/a} \right)^a = \frac{1 - y}{y}.
  \]

- Then, we compute
  \[
  \lambda^a \cdot x^a = \lambda^a \cdot \frac{1 - y}{y} = \frac{\lambda^a - \lambda^a \cdot y^a}{y}.
  \]

- After that, we compute
  \[
  1 + \lambda^a \cdot x^a = 1 + \frac{\lambda^a - \lambda^a \cdot y^a}{y}.
  \]

To simplify this expression, we can add the two fractions in the right-hand side, and get

\[
1 + \lambda^a \cdot x^a = \frac{y + \lambda^a - \lambda^a \cdot y^a}{y} = \frac{\lambda^a + (1 - \lambda^a) \cdot y}{y}.
\]

- Finally, we compute
  \[
  Y = \frac{y}{\lambda^a + (1 - \lambda^a) \cdot y}.
  \]

We indeed get the expression for $Y$ as a ratio of two linear functions of $y$. So, $Y$ can indeed obtained from $y$ by a fractional-linear transformation.

**Comment.** This derivation can be simplified if we take into account that there is no need to first find transform $x^a$ into $x$ and then again compute $x^a$. So, first, we can somewhat simplify the relation between $x$ and $y$ if we take inverses of both sides, then we get

\[
\frac{1}{y} = 1 + x^a.
\]

To separate the variables, we subtract 1 from both sides, resulting in

\[
x^a = \frac{1}{y} - 1.
\]
To simplify this expression, we can add the two fractions in the right-hand side, and get
\[ x^a = \frac{1 - y}{y}. \]

Then, we compute
\[ \lambda^a \cdot x^a = \lambda^a \cdot \frac{1 - y}{y} = \frac{\lambda^a - \lambda^a \cdot y^a}{y}. \]

After that, we compute
\[ 1 + \lambda^a \cdot x^a = 1 + \frac{\lambda^a - \lambda^a \cdot y^a}{y}. \]

To simplify this expression, we can add the two fractions in the right-hand side, and get
\[ 1 + \lambda^a \cdot x^a = \frac{y + \lambda^a - \lambda^a \cdot y^a}{y} = \frac{\lambda^a + (1 - \lambda^a) \cdot y}{y}. \]

Finally, we compute
\[ Y = \frac{y}{\lambda^a + (1 - \lambda^a) \cdot y}. \]