Solution to Homework 1

Definition. *A function $f(x)$ is called* scale-shift-invariant *if for every $\lambda > 0$, there exists a value y_0 such that $y = f(x)$ implies $y' = f(x')$, where we denoted $y' = y + y_0$ and $x' = \lambda \cdot x$.*

Proposition. *If a differentiable function $f(x)$ is scale-shift-invariant, then it is equal to $f(x) = A + a \cdot \ln(x)$ for some A and a.*

Proof. Let us assume that the differentiable function $f(x)$ is scale-shift-invariant. By definition of scale-shift-invariance, this means that for every $\lambda > 0$, there exists some value y_0 (depending on λ) for which $y = f(x)$ implies that $y' = f(x')$, where $y' = y + y_0$ and $x' = \lambda \cdot x$. Since y_0 depends in λ, let us write this dependence in explicit form $y_0 = y_0(\lambda)$.

Let us take any x and take $y = f(x)$. Then, for each λ, we have $y' = f(x')$, where $y' = y + y_0(\lambda)$ and $x' = \lambda \cdot x$. Substituting these expressions for y' and x' into the formula $y' = f(x')$, we conclude that

$$y + y_0(\lambda) = f(\lambda \cdot x).$$

Here, by our choice of y, we have $y = f(x)$. Substituting $f(x)$ instead of y into the above equality, we get

$$f(x) + y_0(\lambda) = f(\lambda \cdot x).$$

Let us now differentiate both sides of this equality with respect to λ: since the functions are equal, their derivatives should be equal too.

With respect to λ, the term $f(x)$ – that does not depend on λ – is a constant. Thus, the derivative of the left-hand side takes the form

$$\frac{dy_0}{d\lambda}.$$

To compute the derivative of the right-hand side, we use the chain rule, and get

$$\frac{d}{d\lambda} f(\lambda \cdot x) = \frac{df}{dx}(\lambda \cdot x) \cdot \frac{d}{d\lambda}(\lambda \cdot x) = \frac{df}{dx}(\lambda \cdot x) \cdot x.$$

Thus, the equality between derivatives of the left-hand side and of the right-hand side takes the form

$$\frac{dy_0}{d\lambda} = \frac{df}{dx}(\lambda \cdot x) \cdot x.$$

This equality is true for every $\lambda > 0$. To simplify this equality, let us take $\lambda = 1$, and let us denote by a the value of the derivative $\frac{dy_0}{d\lambda}$ for $\lambda = 1$. Then, we get the following:

$$a = \frac{df}{dx} \cdot x.$$

Let us separate the variables. For this, we divide both sides of this equality by x and multiply both sides by dx. Then, we get:

$$a \cdot \frac{dx}{x} = df.$$

Now, we integrate both sides, and get

$$a \cdot \ln(x) + A = f,$$

where A is the integration constant. The proposition is proven.