Solution to Homework 6

Task. Prove that the family of all cubic polynomials

$$c_0 + c_1 \cdot x + c_2 \cdot x^2 + c_3 \cdot x^3$$

is invariant with respect to shifts $x \mapsto x + x_0$ and scalings $x \mapsto \lambda \cdot x$, i.e., that if we substitute $x' = x + x_0$ or $x' \lambda \cdot x$,into this expression, we still get a cubic polynomial – with different coefficients x'_i.

Solution. Let us first show shift-invariance. Here,

$$c_0 + c_1 \cdot x' + c_2 \cdot (x')^2 + c_3 \cdot (x')^3 = c_0 + c_1 \cdot (x + x_0) + c_2 \cdot (x + x_0)^2 + c_3 \cdot (x + x_0)^3 =$$

$$c_0 + c_1 \cdot x + c_1 \cdot x_0 + c_2 \cdot (x^2 + 2x \cdot x_0 + x_0^2) + c_3 \cdot (x^3 + 3x^2 \cdot x_0 + 3x \cdot x_0^2 + x_0^3) =$$

$$c_0 + c_1 \cdot x + c_1 \cdot x_0 + c_2 \cdot x^2 + c_2 \cdot 2x \cdot x_0 + c_2 \cdot x_0^2 + c_3 \cdot x^3 + c_3 \cdot 3x^2 \cdot x_0 + c_3 \cdot 3x \cdot x_0^2 + c_3 \cdot x_0^3,$$

i.e., the form

$$c'_0 + c'_1 \cdot x + c'_2 \cdot x^2 + c'_3 \cdot x^3,$$

where we denoted

$$c'_0 = c_0 + c_1 \cdot x_0 + c_2 \cdot x_0^2 + c_3 \cdot x_0^3,$$

$$c'_1 = c_1 + c_2 \cdot 2x \cdot x_0 + c_2 \cdot 2x_0 + c_3 \cdot 3x_0^2,$$

$$c'_2 = c_2 \cdot x^2 + c_3 \cdot 3x_0,$$

and

$$c'_3 = c_3.$$

Let us now show scale-invariance. Here,

$$c_0 + c_1 \cdot x' + c_2 \cdot (x')^2 + c_3 \cdot (x')^3 = c_0 + c_1 \cdot (\lambda \cdot x) + c_2 \cdot (\lambda \cdot x)^2 + c_3 \cdot (\lambda \cdot x)^3.$$

We can now use the fact that $(A \cdot B)^c = A^c \cdot B^c$ to get

$$c_0 + c_1 \cdot \lambda \cdot x + c_2 \cdot \lambda^2 \cdot x^2 + c_3 \cdot \lambda^3 \cdot x^3 = c_0 + (c_1 \cdot \lambda) \cdot x + (c_2 \cdot \lambda^2) \cdot x^2 + (c_3 \cdot \lambda^3) \cdot x^3,$$

i.e., we get

$$c'_0 + c'_1 \cdot x + c'_2 \cdot x^2 + c'_3 \cdot x^3,$$

where $c'_0 = c_0$, $c'_1 = c_1 \cdot \lambda$, $c'_2 = c_2 \cdot \lambda^2$, and $c'_3 = c_3 \cdot \lambda^3$.