Solution to Homework 7

**Task.** As alternatives, let us consider families of the type \( \{ C \cdot f(x) \}_C \), where \( f(x) \) is fixed and \( C \) can take any value. Let us define scaling \( T_\lambda \) as an operation that transforms a family \( \{ C \cdot f(x) \}_C \) into a new family \( \{ C \cdot f(\lambda \cdot x) \}_C \). Prove that if an optimality criterion on the set of all such alternatives is final and scale-invariant, then each function which from the optimal family is a power law \( f(x) = A \cdot x^a \).

*Hint:* first, follow the general proof that we had in class about the function optimal with respect to a \( T \)-invariant criterion, and then use the result that we proved in class – that every scale-scale-invariant function is described by the power law.

**Solution.** Let \( (<, \sim) \) be a final scale-invariant optimality criterion on the set of all alternatives, and let \( F_{\text{opt}} = \{ C \cdot f_{\text{opt}}(x) \}_C \) be the optimal family.

By definition of optimality, this means that \( F_{\text{opt}} \) is better or of the same quality than all other families, i.e., that for every family \( F \), we have:

\[
\text{either } F_{\text{opt}} > F \text{ or } F_{\text{opt}} \sim F.
\]

In particular, this means that for every \( F \) and for every \( \lambda \), we have:

\[
F_{\text{opt}} > T_{1/\lambda}(F) \text{ or } F_{\text{opt}} \sim T_{1/\lambda}(F).
\]

Due to scale-invariance, we have

\[
T_\lambda(F_{\text{opt}}) > T_\lambda(T_{1/\lambda}(F)) \text{ or } T_\lambda(F_{\text{opt}}) \sim T_\lambda(T_{1/\lambda}(F)).
\]

But here, \( T_\lambda(T_{1/\lambda}(F)) = F \). Thus, for every \( F \), we have

\[
T_\lambda(F_{\text{opt}}) > F \text{ or } T_\lambda(F_{\text{opt}}) \sim F.
\]

By definition of an optimal alternative, this means that the family \( T_\lambda(F_{\text{opt}}) \) is optimal. The optimality criterion is final, which means that there is only one optimal family. Thus, \( T_\lambda(F_{\text{opt}}) = F_{\text{opt}}. \)

This means that every function from the family \( T_\lambda(F_{\text{opt}}) = \{ C \cdot f_{\text{opt}}(\lambda \cdot x) \}_C \) also belongs to the family \( F_{\text{opt}}, \text{i.e., has the form } C \cdot f_{\text{opt}}(x) \text{ for some } C. \) In particular, this is true for the function \( f_{\text{opt}}(\lambda \cdot x) \) from the family \( T_\lambda(F_{\text{opt}}) \). Thus, for every \( \lambda \), there exists a value \( C \) depending on \( \lambda \) for which

\[
f_{\text{opt}}(\lambda \cdot x) = C(\lambda) \cdot f_{\text{opt}}(x).
\]

We have proven that any function with this property is a power law.

The statement is proven.