Solution to Homework 7

Task. As alternatives, let us consider families of the type $\{C \cdot f(x)\}_C$, where $f(x)$ is fixed and C can take any value. Let us define scaling T_λ as an operation that transforms a family $\{C \cdot f(x)\}_C$ into a new family $\{C \cdot f(\lambda \cdot x)\}_C$. Prove that if an optimality criterion on the set of all such alternatives is final and scale-invariant, then each function which from the optimal family is a power law $f(x) = A \cdot x^a$.

Hint: first, follow the general proof that we had in class about the function optimal with respect to a T-invariant criterion, and then use the result that we proved in class – that every scale-scale-invariant function is described by the power law.

Solution. Let $(<, \sim)$ be a final scale-invariant optimality criterion on the set of all alternatives, and let $F_{\rm opt} = \{C \cdot f_{\rm opt}(x)\}_C$ be the optimal family.

By definition of optimality, this means that $F_{\rm opt}$ is better or of the same quality than all other families, i.e., that for every family F, we have:

$$\text{either } F_{\rm opt} > F \text{ or } F_{\rm opt} \sim F.$$

In particular, this means that for every F and for every λ, we have

$$F_{\rm opt} > T_{1/\lambda}(F) \text{ or } F_{\rm opt} \sim T_{1/\lambda}(F).$$

Due to scale-invariance, we have

$$T_\lambda(F_{\rm opt}) > T_\lambda(T_{1/\lambda}(F)) \text{ or } T_\lambda(F_{\rm opt}) \sim T_\lambda(T_{1/\lambda}(F)).$$

But here, $T_\lambda(T_{1/\lambda}(F)) = F$. Thus, for every F, we have

$$T_\lambda(F_{\rm opt}) > F \text{ or } T_\lambda(F_{\rm opt}) \sim F.$$

By definition of an optimal alternative, this means that the family $T_\lambda(F_{\rm opt})$ is optimal. The optimality criterion is final, which means that there is only one optimal family. Thus, $T_\lambda(F_{\rm opt}) = F_{\rm opt}$.

This means that every function from the family $T_\lambda(F_{\rm opt}) = \{C \cdot f_{\rm opt}(\lambda \cdot x)\}_C$ also belongs to the family $F_{\rm opt}$, i.e., has the form $C \cdot f_{\rm opt}(x)$ for some C. In particular, this is true for the function $f_{\rm opt}(\lambda \cdot x)$ from the family $T_\lambda(F_{\rm opt})$. Thus, for every λ, there exists a value C depending on λ for which

$$f_{\rm opt}(\lambda \cdot x) = C(\lambda) \cdot f_{\rm opt}(x).$$

We have proven that any function with this property is a power law.

The statement is proven.