Solution to Problem 9

Task. Use what we learned so far to explain the following two empirical formulas:

- $y = x^2 \cdot \exp(-k \cdot x)$; and
- $y = x^{0.3} \cdot \ln(x)$.

Solution. In class, we have learned that if we consider final optimality criteria on the family of all the functions of the type

$$C_1 \cdot e_1(x) + \ldots + C_K \cdot e_K(x),$$

where the functions $e_1(x), \ldots, e_K(x)$ are fixed and the coefficients $C_1, \ldots, C_K$ can take all possible real values, then:

- if the criterion is shift-invariant, then every function from the optimal family is a linear combination of functions of the type

  $$x^p \cdot \exp(a \cdot x) \cdot \sin(b \cdot x) \text{ and } x^p \cdot \exp(a \cdot x) \cdot \cos(b \cdot x),$$

  where p is a non-negative integer and a and b are real numbers;

- if the criterion is scale-invariant, then every function from the optimal family is a linear combination of functions of the type

  $$(\ln(x))^p \cdot x^a \cdot \sin(b \cdot \ln(x)) \text{ and } (\ln(x))^p \cdot x^a \cdot \cos(b \cdot \ln(x)),$$

  where p is a non-negative integer and a and b are real numbers;

- if the criterion is both shift-invariant and scale-invariant, then every function from the optimal family is a polynomial.

Here:

- The function $x^2 \cdot \exp(-k \cdot x)$ is a particular case of the shift-invariant expression $x^p \cdot \exp(a \cdot x) \cdot \cos(b \cdot x)$ corresponding to $p = 2$, $a = -k$, and $b = 0$, so it can be explained by shift-invariance: it is a particular case of functions which are invariant under a shift-invariant optimality criterion.
- The function $x^{0.3} \cdot \ln(x)$ is a particular case of the scale-invariant expression $(\ln(x))^p \cdot x^a \cdot \cos(b \cdot \ln(x))$ corresponding to $p = 1$, $a = 0.3$, and $b = 0$, so it can be explained by scale-invariance: it is a particular case of functions which are invariant under a scale-invariant optimality criterion.