Proof that We Had in Class on September 15

Let us consider families of the type $\{C \cdot f(x)\}_C$, where $f(x)$ is fixed and C can take any value. Let us define shift T_{x_0} as an operation that transforms a family $\{C \cdot f(x)\}_C$ into a new family $\{C \cdot f(x+x_0)\}_C$.

Theorem. *If an optimality criterion on the set of all such alternatives is final and shift-invariant, then each function which from the optimal family has the form* $f(x) = A \cdot \exp(a \cdot x)$.

Proof. Let $(<, \sim)$ be a final shift-invariant optimality criterion on the set of all alternatives, and let $F_{\rm opt} = \{C \cdot f_{\rm opt}(x)\}_C$ be the optimal family.

By definition of optimality, this means that $F_{\rm opt}$ is better or of the same quality than all other families, i.e., that for every family F, we have:

$$\text{either } F_{\rm opt} > F \text{ or } F_{\rm opt} \sim F.$$

In particular, this means that for every F and for every x_0, we have

$$F_{\rm opt} > T_{-x_0}(F) \text{ or } F_{\rm opt} \sim T_{-x_0}(F).$$

Due to scale-invariance, we have

$$T_{x_0}(F_{\rm opt}) > T_{x_0}(T_{-x_0}(F)) \text{ or } T_{x_0}(F_{\rm opt}) \sim T_{x_0}(T_{-x_0}(F)).$$

But here, $T_{x_0}(T_{-x_0}(F)) = F$. Thus, for every F, we have

$$T_{x_0}(F_{\rm opt}) > F \text{ or } T_{x_0}(F_{\rm opt}) \sim F.$$

By definition of an optimal alternative, this means that the family $T_{x_0}(F_{\rm opt})$ is optimal. The optimality criterion is final, which means that there is only one optimal family. Thus, $T_{x_0}(F_{\rm opt}) = F_{\rm opt}$.

This means that every function from the family

$$T_{x_0}(F_{\rm opt}) = \{C \cdot f_{\rm opt}(x+x_0)\}_C$$

also belongs to the family $F_{\rm opt}$, i.e., has the form $C \cdot f_{\rm opt}(x)$ for some C. In particular, this is true for the function $f_{\rm opt}(x+x_0)$ from the family $T_{x_0}(F_{\rm opt})$. Thus, for every x_0, there exists a value C depending on x_0 for which

$$f_{\rm opt}(x+x_0) = C(x_0) \cdot f_{\rm opt}(x).$$

We have already proven that any function with this property has the form $f(x) = A \cdot \exp(a \cdot x)$.

The statement is proven.