Let us consider families of the type \( \{ C \cdot f(x) \}_C \), where \( f(x) \) is fixed and \( C \) can take any value. Let us define shift \( T_{x_0} \) as an operation that transforms a family \( \{ C \cdot f(x) \}_C \) into a new family \( \{ C \cdot f(x + x_0) \}_C \).

**Theorem.** If an optimality criterion on the set of all such alternatives is final and shift-invariant, then each function which from the optimal family has the form \( f(x) = A \cdot \exp(a \cdot x) \).

**Proof.** Let \((<, \sim)\) be a final shift-invariant optimality criterion on the set of all alternatives, and let \( F_{\text{opt}} = \{ C \cdot f_{\text{opt}}(x) \}_C \) be the optimal family.

By definition of optimality, this means that \( F_{\text{opt}} \) is better or of the same quality than all other families, i.e., that for every family \( F \), we have:

either \( F_{\text{opt}} > F \) or \( F_{\text{opt}} \sim F \).

In particular, this means that for every \( F \) and for every \( x_0 \), we have

\[ F_{\text{opt}} > T_{-x_0}(F) \text{ or } F_{\text{opt}} \sim T_{-x_0}(F). \]

Due to scale-invariance, we have

\[ T_{x_0}(F_{\text{opt}}) > T_{x_0}(T_{-x_0}(F)) \text{ or } T_{x_0}(F_{\text{opt}}) \sim T_{x_0}(T_{-x_0}(F)). \]

But here, \( T_{x_0}(T_{-x_0}(F)) = F \). Thus, for every \( F \), we have

\[ T_{x_0}(F_{\text{opt}}) > F \text{ or } T_{x_0}(F_{\text{opt}}) \sim F. \]

By definition of an optimal alternative, this means that the family \( T_{x_0}(F_{\text{opt}}) \) is optimal. The optimality criterion is final, which means that there is only one optimal family. Thus, \( T_{x_0}(F_{\text{opt}}) = F_{\text{opt}} \).

This means that every function from the family

\[ T_{x_0}(F_{\text{opt}}) = \{ C \cdot f_{\text{opt}}(x + x_0) \}_C \]

also belongs to the family \( F_{\text{opt}} \), i.e., has the form \( C \cdot f_{\text{opt}}(x) \) for some \( C \). In particular, this is true for the function \( f_{\text{opt}}(x + x_0) \) from the family \( T_{x_0}(F_{\text{opt}}) \). Thus, for every \( x_0 \), there exists a value \( C \) depending on \( x_0 \) for which

\[ f_{\text{opt}}(x + x_0) = C(x_0) \cdot f_{\text{opt}}(x). \]

We have already proven that any function with this property has the form \( f(x) = A \cdot \exp(a \cdot x) \).

The statement is proven.