Test 1

Problem 1. A function $f(x)$ is called *shift-scale-invariant* if for every x_0, there exists a value μ such that $y = f(x)$ implies $y' = f(x')$, where we denoted $y' = \mu \cdot y$ and $x' = x + x_0$. Prove that every differentiable shift-scale invariant function has the form $f(x) = A \cdot \exp(a \cdot x)$ for some A and a.

Problem 2. Suppose that you know the values of some quantity v in two points x_1 and x_2, these values are $v_1 = 100$ and $v_2 = 200$. Based on this information, we want to use the inverse distance weighting technique to predict the value v of this quantity as a point x for which $d(x_1, x) = 10$ and $d(x, x_2) = 20$. Take $a = -1$.

Reminder. The general formula has the form

$$v = \frac{\sum_i v_i \cdot (d(x, x_i))^a}{\sum_i (d(x, x_i))^a}.$$

Problem 3. Use calculus to find the value x for which the following function attains is minimum $2x^2 - 3x + 1$. What is the value of this minimum?

Problem 4. Describe a function $y = 1/(1 + x^2)$ as a composition of invariant functions.

Comment. This function is actively used in physics and in uncertainty quantification.

Problem 5. Describe a function $\sqrt[3]{x^3 + x^6}$ as a scale-invariant combination of two scale-scale-invariant functions.

Hint: use the fact that $x^6 = (x^2)^3$.

Problem 6. Prove that the family of all linear functions

$$c_0 + c_1 \cdot x$$

is invariant with respect to shifts $x \mapsto x + x_0$ and scalings $x \mapsto \lambda \cdot x$, i.e., that if we substitute $x' = x + x_0$ or $x' = \lambda \cdot x$,into this expression, we still get a linear function – with different coefficients c_i'.

Problem 7. As alternatives, let us consider families of the type $\{C \cdot f(x)\}_C$, where $f(x)$ is fixed and C can take any value. Let us define shift T_{x_0} as an operation that transforms a family $\{C\cdot f(x)\}_C$ into a new family $\{C\cdot f(x+x_0)\}_C$. Prove that if an optimality criterion on the set of all such alternatives is final and shift-invariant, then each function which from the optimal family has the form $f(x) = A \cdot \exp(a \cdot x)$.

Problem 8. Describe your progress on the class project.

Problem 9. Briefly explain what is the purpose of this class.