Solutions to Test 2

1. Why do we sometimes need to consider families of functions instead of a single function?

Answer. In some cases, the dependence $y = f(x)$ between the two quantities x and y is exactly the same in all situations. This happens, for example, in physics. However, in other cases, the dependence is somewhat different in different situations. In such cases, we cannot use the same function to describe all these dependencies, we need to have a family of functions.

2. What type of families do we consider? Provide an example of such a family.

Answer. We consider families of the type $C_1 \cdot e_1(x) + \ldots + C_n \cdot e_n(x)$, where the functions $e_1(x)$, $\ldots$, $e_n(x)$ are fixed, and the coefficients C_1, $\ldots$, C_n can be arbitrary real numbers. An example if all polynomial of given order. This family appears when we take $e_1(x) = 1$, $e_2(x) = x$, $e_3(x) = x^2$, etc.

3–5. Explain, for a family of functions:

- how shift-invariance leads to functional equations,
- how functional equations leads to a system of differential equations, and
- what is the general solution to this system of differential equations.

Answer. Shift-invariance means that for every function $f(x)$ from the family, a shifted function $f(x + x_0)$ also belongs to this family. In particular, each of the functions $e_i(x)$ belongs to the family – it corresponds to $C_i = 1$ and $C_j = 0$ for all $j \neq i$. Thus, the shifted function $e_i(x + x_0)$ should also belong to the family. This means that this function can be represented as

$$e_i(x + x_0) = C_{i1}(x_0) \cdot e_1(x) + \ldots + C_{in}(x_0) \cdot e_n(x)$$

for some coefficients C_{ij} that depend on x_0.

If we differentiate both sides of this functional equation by x_0, we get

$$e_i'(x + x_0) = C_{i1}'(x_0) \cdot e_1(x) + \ldots + C_{in}'(x_0) \cdot e_n(x).$$

In particular, for $x_0 = 0$, we get

$$e_i'(x) = c_{i1} \cdot e_1(x) + \ldots + c_{in} \cdot e_n(x),$$

where c_{ij} denotes $c_{ij} \stackrel{\text{def}}{=} C'_{ij}(0)$.

This system of differential equations is known as the system of linear differential equations with constant coefficients. It is known that every solution to this system is a linear combination of expressions $x^m \cdot \exp(k \cdot x)$, where k is an eigenvalue of the matrix c_{ij}m and m is a non-negative integer which is smaller than the multiplicity of this eigenvalue. This eigenvalue is, in general, a complex number.

6. What is the general form of functions from a scale-invariant family? Explain how this leads to a general form of functions from a family which is both shift-invariant and scale-invariant.

Answer. A general function from a scale-invariant family is a linear combination of expressions $(\ln(x))^m \cdot x^k$. If a family is both shift-invariant and scale-invariant, then:

- it cannot contain exp terms, since scale-invariant expressions do not have them, and
- it cannot contain ln terms, since shift-invariant expressions do not have them.

Thus, it can only contain terms of the type x^m, where m is a non-negative integer. Linear combinations of such terms are polynomials.

7. How can you explain the following empirical formula: $x^2 \cdot (\ln(x))^2$?

Answer. This is a particular case of a function from a scale-invariant family.

8. Explain what is a transformation group, and why the class of all natural transformations should be a finite-parametric transformation group that contains all linear transformations. Prove that the class of all linear functions is a transformation group.

Answer. Linear transformations are natural: $x \to a \cdot x$ corresponds to changing the measuring unit, $x \to x + b$ corresponds to changing the starting point, and $x \to a \cdot x + b$ corresponds to changing both. If we have a natural transformation from scale A to scale B, then the inverse transformation – from scale B to scale A – should also be natural.

If we have two natural transformations, then applying one after another – i.e., considering a composition of these transformations – should also be a natural transformation.

A class of functions that contains the inverse of every function from this class and the composition of every two functions from this class is called a transformation group. Thus, the class of all natural transformations should be a transformation group.

We want to represent all these functions in a computer, and in each computer, there is a limit on how many numbers we can store. Thus, we need a family that depends on finitely many parameters.

For a linear transformation $y = a \cdot x + b$, we can obtain the inverse if we find how x depends on y, i.e., if we solve this linear equation with x as an unknown. Then, we get $y - b = a \cdot x$ and thus, $x = (1/a) \cdot y - b/a$, i.e., also a linear function.

If we take $y = a \cdot x + b$ and $z = A \cdot y = B$, then we get

$$z = A \cdot (a \cdot x + b) + B = A \cdot a \cdot x + A \cdot b + B = (A \cdot a) \cdot x + (A \cdot b + B),$$

also a linear function.

9. Describe all functions from a finite-parametric transformation group that contain all linear transformations.

Answer. All such functions are fractional linear, i.e., ratios of two linear functions

$$f(x) = \frac{a \cdot x + b}{c \cdot x + d}.$$

10. Explain the following empirical dependences:

$$y = \frac{x}{1 + x} \text{ and } y = \frac{1}{1 + \exp(-x)}.$$

Answer. The first one is a fractional linear function, so it presents a natural transformation between the quantities x and y.

The second transformation comes from the requirement that if we shift x, i.e., replace x with $X = x + x_0$, then the new value $Y = f(X)$ should be related to the original value $y = f(x)$ by a natural transformation. In other words, this is an example of shift-natural invariance: for each x_0 there exits value a, b, c, and d depending on x_0 for which

$$Y = \frac{a(x_0) \cdot y + b(x_0)}{c(x_0) \cdot y + d(x_0)},$$

where $y = f(x)$, $Y = f(X)$, and $X = x + x_0$.