New Ideas about Joint Inversion (as described in a recent paper on spline techniques)

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Main Objective of Our . . .
Gravity . . .
Normal Mode . . .
Traditional Approach . . .
First Idea: in Brief
Details: A Different Basis
A Related Open Problem
Second Idea: In Brief

1. Main Objective of Our Presentation

- *Paper:* Paula Berkel, Doreen Fischer, and Volker Michel, "Spline multiresolution and numerical results for joint gravitation and normal-mode inversion with an outlook on sparse regularisation", *International Journal of Geomathematics*, August 2010.
- *Main objective of the paper:* use joint inversion to provide a full 3-D picture of the Earth.
- *Geophysically:* the authors combine gravity and normal-mode measurements.
- *Our applications:* we usually deal with a *local* geophysical analysis.
- *Our objective:* to describe the main idea that can be used in our applications.

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2. Gravity Measurements: Non-Uniqueness

- *Ideally:* once we know the gravitational potential field $\varphi(x)$, we can determine the density $\rho(x)$:

$$\rho = \text{const} \cdot \nabla^2 \varphi(x).$$

- *In practice:*
  - we only know the values of $\varphi(x)$ with measurement errors;
  - we only know the values $\varphi(x)$ in some points x – all of which are outside the Earth.
- *Result:* we cannot uniquely reconstruct the density $\rho(x)$ ("Earth model") from the gravity measurement results.
- *Specifically:* several different Earth models $\rho(x)$ are consistent with the same measurement results.

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3. Non-Uniqueness (cont-d)

- *Additional problem:* in the isotropic case, the gravity outside the Earth is $\sim \dfrac{M}{R^2}$
  - this is true when the mass is uniformly distributed inside the Earth;
  - this is true when the mass is mostly concentrated in the center;
  - etc.
- *Conclusion:*
  - based on measured gravity values,
  - we cannot uniquely determine how density is distributed inside the Earth.
- Thus, to determine an Earth model, we need to supplement gravity data with *other measurements*.

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4. Need for Error Bounds and for Faster Computations

- *Need for error bounds:*
  - measurement errors cause errors in the resulting Earth model;
  - it is desirable to find the error bounds on the parameters of the resulting Earth model.
- *Need for faster computations:*
  - in many data processing techniques, we get values on a dense grid;
  - it often turns out that spatial resolution is not so high – especially at depth;
  - this means that the difference between $\rho(x)$ at two neighboring points is not statistically meaningful;
  - it is desirable to save computation time and only generate meaningful values.

5. Normal Mode Measurements

- Gravity measurements provide info about deep layers of Earth.
- To supplement this info, we need to use other geophysical info about such deep layers.
- Such info comes after a strong earthquake, when
  - not only a seismic wave reaches a station,
  - but also it reaches practically the whole Earth and starts oscillations that go on for some time.
- Such post-earthquake oscillations are called *normal mode* oscillations.
- In the paper, gravity measurements are combined with the normal mode measurements.

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6. Traditional Approach to Analyzing Seismic Data

- To understand the new ideas, let us recall the traditional approach.
- First, we reconstruct the velocity (or, equivalently) density model in each 3-D location.
- The model coming out of the computer program has some "features", e.g., areas where:
  - either density is higher than around them,
  - or density is lower than around them.
- Some of these features are real.
- Some are artifacts of the method – caused by uncertainty and incompleteness of data.
- One of the ways to distinguish between real features and artifacts is to use a checkerboard method.

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7. Traditional Approach (cont-d)

- *Checkerboard method* – main idea:
  - we add sinusoidal "checkerboard" patterns to the model,
  - we simulate measurement results corresponding to this perturbed model, and
  - we apply the algorithm to the simulated measurement results.
- *Case 1:* the algorithm detects the perturbations of this spatial size.
- *Conclusion:* features of this size in the original model were real.
- *Case 2:* the algorithm does not detect perturbations.
- *Conclusion:* features of this small size are artifacts.

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8. First Idea: in Brief

- *Problem with the traditional approach:*
  - we produce a large number of values
  - and then we, in effect, dismiss many of these values as artifacts.
- *Fact:* an arbitrary function can be approximated by Fourier series

$$\rho(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\cdot\sin(i\cdot x\cdot\omega_0+j\cdot y\cdot\omega_0+\varphi).$$

- *Traditional approach, in effect:* find all the values a_{ij}, then dismiss most of them.
- *New idea:* only find the values a_{ij} that can be (statistically reliably) reconstructed.
- *How:* we generate a_{ij} one by one until we get too large a reconstruction error.

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9. Details: A Different Basis

- *In general:*
  - we select a basis $e_0(x)$, $e_1(x)$, etc., and
  - we represent an arbitrary function $f(x)$ as a linear combination of the basis functions $e_n(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot e_n(x).$$

- *Examples of bases:*
  - polynomials $1, x, x^2, \ldots,$ resulting in Taylor series;
  - sines, resulting in Fourier series.
- *In the above method:* we use sines as the basis.
- *In general:* other bases are possible.
- *The paper:* uses a combination of polynomials and sines.

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10. A Related Open Problem

- *In the proposed method*:
  - if we cannot recover a_{ij} with a good accuracy,
  - we stop and dismiss all the values corresponding to this spatial resolution.
- *Problem:* this approach does not take into account that accuracy depends on depth.
- *In the checkerboard method:* for each spatial frequency,
  - we keep shallower features if they can be recovered from the perturbed data, and
  - we dismiss deeper features if they cannot be recovered from the perturbed data.
- *Open problem:* it is not clear how to do this in the proposed method; maybe use wavelets?

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11. Second Idea: In Brief

- *Gravity force:* an acceleration $\vec a(\vec x)$ caused at location $\vec x$ by n point bodies with masses m_i at locations $\vec r_i$:

$$\vec a(\vec x) = \sum_{i=1}^{n} \frac{G\cdot m_i}{|\vec r_i - \vec x|^3}\cdot(\vec r_i - \vec x).$$

- *Continuous case:*

$$\vec a(\vec x) = G\cdot \int \frac{\rho(\vec r)}{|\vec r - \vec x|^3}\cdot(\vec r - \vec x)\,d\vec r.$$

- *Fact:* this dependence is linear in $\rho(\vec r)$.
- Once we select a basis $e_{ij}(\vec r)$, for $\rho(\vec r) = \sum\limits_{i,j} a_{ij}\cdot e_{ij}(\vec r)$, we get

$$\vec a(\vec x) = G\cdot \sum_{i,j} a_{ij}\cdot \int \frac{e_{ij}(\vec r)}{|\vec r - \vec x|^3}\cdot(\vec r - \vec x)\,d\vec r.$$

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12. Second Idea (cont-d)

- *We want to know:* how the acceleration $\vec{a}(\vec{x})$ at a location $\vec{x}$ is related to the unknown density $\rho(\vec{r})$.
- *Answer (reminder):* $\vec{a}(\vec{x}) = \sum\limits_{i,j} a_{ij} \cdot I_{ij}$, where

$$I_{ij} \stackrel{\text{def}}{=} G \cdot \int \frac{e_{ij}(\vec{r})}{|\vec{r}-\vec{x}|^3} \cdot (\vec{r}-\vec{x})\, d\vec{r}.$$

- *Fact:* the values I_{ij} do not depend on observations, and can thus be pre-computed.
- *Resulting idea:*
  - we pre-compute the values I_{ij} before data processing starts;
  - on the data processing stage, we find a_{ij} by simply solving a system of linear equations.

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13. Second Idea: Discussion

- *Advantage:*
  - we pre-compute the auxiliary values I_{ij} once, and
  - then use the pre-computed values I_{ij} to process all the data;
  - thus, we save data processing time.
- *Need to take uncertainty into account*
  - *Fact:* we only measure $\vec{a}$ with some accuracy.
  - *Thus:* we need to use the Least Squares method to solve the resulting system of linear equations

$$\vec{a}(\vec{x}) \approx \sum_{i,j} a_{ij} \cdot I_{ij}.$$

  - *For normal mode* measurements: the authors use a similar idea – based on linearization.

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