Why Some Non-Classical Logics Are More Studied?

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1. Commonsense Reasoning and Formal Logic

- We all use logic in real life.
- We use phrases containing "and", "or", and "not" in our reasoning.
- Since ancient times, researchers have been trying to describe such reasoning in precise terms.
- In other words, they tried to transform commonsense reasoning into formal logic.

2. Traditional Logic: a Brief Reminder

- The most widely used formalization of logic is the traditional 2-valued logic.
- It was formally described by Boole in the 19th century.
- Operations of this logic have many well-known properties.
- For example, for the "and"-operation $a \,\&\, b$ is:
  - "false" (0) and any a is equivalent to "false":
  $$0 \,\&\, a \Leftrightarrow 0;$$
  - similarly, a and "false" is equivalent to "false":
  $$a \,\&\, 0 \Leftrightarrow 0;$$
  - "true" (1) and any a is equivalent to a, as well as a and "true": $1 \,\&\, a \Leftrightarrow a \,\&\, 1 \Leftrightarrow a$.

3. Traditional Logic (cont-d)

- This operation is *idempotent*: $a \,\&\, a \Leftrightarrow a$.
- This operation is *commutative*: $a \,\&\, b \Leftrightarrow b \,\&\, a$.
- This operation is *associative*: $a \,\&\, (b \,\&\, c) \Leftrightarrow (a \,\&\, b) \,\&\, c$.
- Similarly, the "or"-operation $a \vee b$ also satisfies similar properties:
  - $0 \vee a \Leftrightarrow a \vee 0 \Leftrightarrow a$;
  - $1 \vee a \Leftrightarrow a \vee 1 \Leftrightarrow 1$;
  - it is idempotent, i.e., $a \vee a \Leftrightarrow a$;
  - it is commutative, i.e., $a \vee b \Leftrightarrow b \vee a$;
  - it is associative, i.e., $a \vee (b \vee c) \Leftrightarrow (a \vee b) \vee c$.
- These two operations are distributive with respect to each other:

$$a \,\&\, (b \vee c) \Leftrightarrow (a \vee b) \,\&\, (a \vee c); \;\; a \vee (a \,\&\, c) \Leftrightarrow (a \vee b) \,\&\, (a \vee c).$$

4. Traditional Logic (cont-d)

- Many properties of the traditional logic involve negation:
  - the rule that $\neg(1)$ is 0, and that $\neg(1)$ is 0;
  - the law of excluded middle, according to which $a \vee \neg a$ is always true;
  - the double negation rule $\neg\neg a \Leftrightarrow a$; and
  - de Morgan laws:

$$\neg(a \,\&\, b) \Leftrightarrow \neg a \vee \neg b; \quad \neg(a \vee b) \Leftrightarrow \neg a \,\&\, \neg b.$$

5. The Traditional 2-Valued Logic Is Only an Approximation to Commonsense Reasoning

- It is well known that boolean logic is only an approximation to how we actually reason.
- Our actual use of "and" and "or" is more complex.
- For example:
  - while in the formal logic, "and" is commutative,
  - the phrases "I studied and I took the test" and "I took the test and I studied" are different.
- Because of this difference, researchers have been trying to come up with:
  - extensions and modification of boolean logic
  - that would better capture commonsense reasoning.
- As a result, we have a plethora of different non-classical logics.

6. Non-Classical Logics: Examples

- Since the early 20th century, many extensions and generalizations of classical logic have appeared.
- In some of these logics, propositional operations "and", "or", and "not" do not satisfy all the usual properties.
- Other logics introduce additional propositional operations.
- Yet other logics do both.
- One of the first examples of non-classical logics was *intuitionistic logic* – developed early in the 20th century.
- This logic rejects the law of excluded middle, the law of double negation, and de Morgan laws.

7. Non-Classical Logics (cont-d)

- Another example of a non-classical logic is *modal logic*.
- This logic introduces additional unary operations "necessary" and "possible".
- This logic was known since Aristotle but it was formalized only in the early 20th century.
- *Linear logic* rejects the rule $a \,\&\, a \Leftrightarrow a$.
- Some logics use two different negation operations – usual negation and strong negation, etc.

8. Challenge

- But why some extensions were developed, and some were not?
- For example, as we have mentioned earlier, in commonsense reasoning, "and" is not always commutative.
- However, no mainstream logics seriously considers non-commutative "and"-operations.
- So why there have been developed logics rejecting some laws of boolean logic and not others?

9. What We Do in This Talk

- At first glance, it may seem that the above challenge has no good answer.
- It is like asking why Picasso moved to a blue period and not to some other period.
- However, surprisingly, we do present an answer.
- To come up with such an answer, we take into account yet another non-classical logic: fuzzy logic.

10. What We Do in This Talk (cont-d)

- Fuzzy logic was developed specifically to capture important features of commonsense reasoning.
- In fuzzy logic:
  - in addition to the traditional two truth values "true" (1) and "false" (0),
  - we also allow intermediate truth values which are represented by numbers from the interval $[0, 1]$.
- It turns out that fuzzy ideas can indeed explain:
  - why some extensions of boolean logic have indeed been well-studied
  - and why some extensions have not yet been thoroughly explored.

11. Background

- In fuzzy logic, we extend propositional operations from the 2-valued set $\{0, 1\}$ to the interval $[0, 1]$.
- Then each operation becomes a continuous function of the corresponding real-valued variables.
- In general, such functions can be expanded in Taylor series – and can be thus approximated by polynomials.
- This corresponds to keeping first few terms in this expansion.
- As we increase the order of the corresponding polynomials:
  - we get more and more accurate representation of the corresponding functions,
  - but at the same time this representation becomes more complex.

12. Our Main Idea

- It is natural to expect that:
  - violations of logical laws that can be attained by the simplest functions will be explored first, while
  - violations that require much higher (and thus, more complex) polynomials will be studied much later.
- We will show that this natural idea helps explain why:
  - some non-classical logics have been thoroughly studied, while
  - others remain largely a not-well-studied idea.

13. Let Us Start With “And”

- Let us consider the very first properties of an “and”-operation.
- Based on these properties, we have the following simple (and known) result.
- **Proposition.**
  - *There is no linear function* $f(a,b)$ *for which* $f(0,a)=f(a,0)=0$ *and*

$$f(1,a)=f(a,1)=a.$$

  - *The only quadratic function with these properties is*

$$f(a,b)=a\cdot b.$$

14. Proof

- A general quadratic function of two variables has the form

$$f(a,b) = c_0 + c_1 \cdot a + c_2 \cdot b + c_{11} \cdot a^2 + c_{12} \cdot a \cdot b + c_{22} \cdot b^2.$$

- The requirement $f(0,a) = 0$ implies that $c_0 + c_2 \cdot a + c_{22} \cdot a^2 = 0$ for all a, hence $c_0 = c_2 = c_{22} = 0$.
- Similarly, the requirement $f(a,0) = 0$ implies that $c_1 = c_{11} = 0$.
- Thus, $f(a,b) = c_{12} \cdot a \cdot b$.
- The condition $f(1,a) = a$ now implies that $c_{12} = 1$ and hence, that $f(a,b) = a \cdot b$.
- The proposition is proven.

15. “Or”-Operations and Negations

- A similar result holds for “or”-operations:
- **Proposition.**
  - *There is no linear function* $f(a,b)$ *for which* $f(0,a) = f(a,0) = a$ *and*

    $$f(1,a) = f(a,1) = 1.$$
  - *The only quadratic function with these properties is* $f(a,b) = a + b - a \cdot b$.
- For negation, a linear operation is possible:
- **Proposition.** *The only linear function* $f(a)$ *for which* $f(0) = 1$ *and* $f(1) = 0$ *is the function* $f(a) = 1 - a$.
- **Proof** is straightforward: it is based on the general form of a linear function of one variable:

  $$f(a) = c_0 + c_1 \cdot a.$$

16. Discussion

- What properties are satisfied for these simple operations:
  - $a \cdot b$ for "and",
  - $a + b - a \cdot b$ for "or", and
  - $1 - a$ for "not"?
- The "and" and "or"-operations are commutative and associative.
- The law of double negation is satisfied.
- de Morgan laws are satisfied.

17. Discussion (cont-d)

- However, already for this simple example, we can see that two major laws are not satisfied:
  - we do not have the excluded middle:

$$a \vee \neg a = 1 + (1 - a) - a \cdot (1 - a) = 1 - a \cdot (1 - a) \neq 1,$$

  - and in general, we have $a \mathbin{\&} a = a^2 \neq a$, so $a \mathbin{\&} a$ is not always equivalent to a.
- Not surprisingly, the logics based on these violations – intuitionistic logic and linear logic – have been studied.

18. General Case of Quadratic Functions

- So far, we considered linear negation operations; however:
  - since the only "and"- and "or"-operations are quadratic anyway,
  - why not consider quadratic negation operations as well?

- **Proposition.** *For quadratic functions $f(a)$, the following two conditions are equivalent to each other:*
  - $f(0) = 1$ *and* $f(1) = 0$, *and*
  - $f(a) = 1 - a - c_{11} \cdot a \cdot (1 - a)$.

19. Proof

- Let us consider a general quadratic function of one variable $f(a) = c_0 + c_1 \cdot a + c_{11} \cdot a^2$.
- The condition $f(0) = 1$ implies that $c_0 = 1$.
- The condition $f(1) = 0$ implies that $1 + c_1 + c_{11} = 0$, i.e., that $c_1 = -1 - c_{11}$.
- Thus, the general expression takes the form

$$f(a) = 1 - a - c_{11} \cdot a + c_{11} \cdot a^2 = 1 - a - c_{11} \cdot a \cdot (1 - a).$$

- The proposition is proven.

20. Discussion

- When $c_{11} \neq 0$, we have $f(f(a)) \neq a$.
- This explains why logics with no double negation property have been actively studied.
- We can have several different such operations, corresponding to different values c_{11}.
- This explains why logics with several different negation operations have been considered.
- The resulting function is not necessarily monotonic.
- This explains why non-monotonic logics have also been actively studied.

21. Other Quadratic Operations

- What are the general extension of the identity function, i.e., a function for which $f(0) = 0$ and $f(1) = 1$?
- **Proposition.**
  - *The only linear function $f(a)$ for which $f(0) = 0$ and $f(1) = 1$ is the trivial function $f(a) = a$.*
  - *There exist non-trivial quadratic functions with the above properties; they all have the form*

$$f(a) = a - c_{11} \cdot a \cdot (1 - a).$$

22. Proof

- Let us consider a general quadratic function

$$f(a) = c_0 + c_1 \cdot a + c_{11} \cdot a^2.$$

- The requirement that $f(0) = 0$ implies that $c_0 = 0$.
- The requirement that $f(1) = 1$ implies that $c_1 + c_{11} = 1$, thus $c_1 = 1 - c_{11}$.
- So, the general expression takes the form

$$f(a) = a - c_{11} \cdot a + c_1 \cdot a^2 = a - c_{11} \cdot a \cdot (1 - a).$$

- The proposition is proven.

23. Discussion

- When $c_{11} > 0$, we have $f(a) \le a$.
- This can be identified with the unary operation "necessary", for which the usual intuition is that:
  - if something is absolutely true, it is also absolutely necessarily true: $f(1) = 1$;
  - if something is absolutely false, it is also absolutely necessarily false: $f(0) = 0$.
- In general, if something is necessarily true, then it is true – but not vice versa, so:
  - our degree of confidence that a statement is necessarily true can be smaller than
  - our degree of confidence that it is true – it could be true accidentally.

24. Discussion (cont-d)

- When $c_{11} < 0$, we get always $x \le f(x)$.
- This can be identified with unary operation "possible" in modal logic.
- So it is not surprising that modal logic have been actively developed.
- It should also be noticed that, in general, for $c_{11} \ne 0$, we have $f(f(a)) \ne a$.
- This explains why modal logic actively studies logics in which:
  - an iteration of necessity
  - is not equivalent to a single necessity operation.

25. What About Cubic Operations?

- Let us consider cubic "and"- and "or"-operations, i.e., functions of the type

$$f(a,b) = c_0 + c_1 \cdot a + c_2 \cdot b + c_{11} \cdot a^2 + c_{12} \cdot a \cdot b + c_{22} \cdot b^2 +$$
$$c_{111} \cdot a^3 + c_{112} \cdot a^2 \cdot b + c_{122} \cdot a \cdot b^2 + c_{222} \cdot b^3.$$

- **Proposition.** *The only cubic function for which $f(0,a) = f(a,0) = 0$ and $f(1,a) = f(a,1) = a$ is $f(a,b) = a \cdot b$.*
- **Proof.** The condition $f(0,b) = 0$ implies that, for all b, we have $c_0 + c_2 \cdot b + c_{22} \cdot b^2 + c_{222} \cdot b^3 = 0$.
- Thus, $c_0 = c_2 = c_{22} = c_{222} = 0$.
- Similarly, the condition $f(a,0) = 0$ implies that $c_1 = c_{11} = c_{111} = 0$.
- Thus, the general expression takes the form

$$f(a,b) = c_{12} \cdot a \cdot b + c_{112} \cdot a^2 \cdot b + c_{122} \cdot a \cdot b^2.$$

26. Cubic Operations (cont-d)

- For this formula, the requirement that $f(1, b) = b$ implies that for all b, we have $c_{12} \cdot b + c_{112} \cdot b + c_{122} \cdot b^2 = 0$.
- Hence $c_{122} = 0$.
- Similarly, the requirement that $f(a, 1) = a$ implies that

$$c_{112} = 0.$$

- Thus, all cubic terms are 0, so $f(a, b)$ is actually a quadratic function.
- For quadratic functions, we already know that the only operation with the desired properties is $f(a, b) = a \cdot b$.
- The proposition is proven.
- **Proposition.** *The only cubic function for which $f(0, a) = f(a, 0) = a$ and $f(1, a) = f(a, 1) = 1$ is*

$$f(a, b) = a + b - a \cdot b.$$

27. Discussion

- So, even if we consider cubic terms, we will still get only commutative and associative "and" and "or".
- This explains why:
  - in the vast majority of logics studied so far,
  - these operations are indeed commutative and associative.
- To find example of non-commutative and/or non-associative logics, we need to go to polynomials of higher orders.

28. Case of 4th Order Operations

- Let us consider general 4th order functions

$$f(a,b)=c_0+c_1\cdot a+c_2\cdot b+c_{11}\cdot a^2+c_{12}\cdot a\cdot b+c_{22}\cdot b^2+$$
$$c_{111}\cdot a^3+c_{112}\cdot a^2\cdot b+c_{122}\cdot a\cdot b^2+c_{222}\cdot b^3+$$
$$c_{1111}\cdot a^4+c_{1112}\cdot a^3\cdot b+c_{1122}\cdot a^2\cdot b^2+c_{1222}\cdot a\cdot b^3+c_{2222}\cdot b^4.$$

- **Proposition.** *For 4th order functions, the following two conditions are equivalent to each other:*
  - *for all* a, *we have* $f(0,a)=f(a,0)=0$ *and* $f(1,a)=f(a,1)=a$, *and*
  - *the function* $f(a,b)$ *has the form*

$$f(a,b)=a\cdot b-c\cdot a\cdot(1-a)\cdot b\cdot(1-b).$$

29. Proof

- The condition $f(0, b) = 0$ implies that for all b, we have $c_0 + c_2 \cdot b + c_{22} \cdot b^2 + c_{222} \cdot b^3 + c_{2222} \cdot b^4 = 0$.
- Thus, $c_0 = c_2 = c_{22} = c_{222} = c_{2222} = 0$.
- Similarly, the condition that $f(a, 0) = 0$ for all a implies that $c_1 = c_{11} = c_{111} = c_{1111} = 0$.
- Thus, the general formula gets the following simplified form: $f(a, b) =$

  $$c_{12} \cdot a \cdot b + c_{112} \cdot a^2 \cdot b + c_{122} \cdot a \cdot b^2 + c_{1112} \cdot a^3 \cdot b + c_{1122} \cdot a^2 \cdot b^2 + c_{1222} \cdot a \cdot b^3.$$
- For this function, the requirement that $f(1, b) = b$ for all b implies that

  $$c_{12} \cdot b + c_{112} \cdot b + c_{122} \cdot b^2 + c_{1112} \cdot b + c_{1122} \cdot b^2 + c_{1222} \cdot b^3 = b.$$
- Thus, $c_{1222} = 0$.
- Similarly, the condition $f(a, 1) = a$ implies $c_{1112} = 0$.

30. Proof (cont-d)

- Thus, the above equality takes the following simplified form: $(c_{12} + c_{112}) \cdot b + (c_{122} + c_{1122}) \cdot b^2 = b$.
- So, $c_{12}+c_{112} = 1$, hence $c_{112} = 1-c_{12}$, and $c_{1112} = -c_{112}$.
- So, if we denote $c \stackrel{\text{def}}{=} 1-c_{12}$, we get $c_{112} = c$, $c_{12} = 1-c$, and $c_{1122} = -c$.
- Similarly, from the condition that $f(a,1) = a$, we conclude that $c_{122} = c$.
- Thus, we get the desired expression for the function:

$$f(a,b) = a \cdot b - c \cdot (a \cdot b - a^2 \cdot b - a \cdot b^2 + a^2 \cdot b^2) =$$

$$a \cdot b \cdot (1 - c \cdot (1 - a - b + a \cdot b)) =$$

$$a \cdot b(1-c\cdot(1-a)\cdot(1-b)) = a\cdot b-c\cdot a\cdot(1-a)\cdot b\cdot(1-b).$$

- The proposition is proven.

31. "Or"-Operations

- For "or"-operations we have a similar result.
- **Proposition.** *For 4th order functions, the following two conditions are equivalent to each other:*
  - *for all* a*, we have* $f(0,a) = f(a,0) = a$ *and* $f(1,a) = f(a,1) = 1$*, and*
  - *the function* $f(a,b)$ *has the form*

$$f(a,b) = a + b - a \cdot b - c \cdot a \cdot (1-a) \cdot b \cdot (1-b).$$

32. Discussion

- We did not assume commutativity or associativity.
- Interestingly, we got operations which are commutative but *not* associative.
- This explains why:
  - some research has been done for non-associative logics,
  - while much fewer results are known for non-commutative ones.
- To get non-commutative operations, we need to consider at least 5th order polynomials.
- For 5th order polynomials, it is already possible to have a non-commutative operation; example: take

$$f(a,b) = a \cdot b - (c_a \cdot a + c_b \cdot b) \cdot a \cdot (1-a) \cdot b \cdot (1-b).$$

- This operation is non-commutative when $c_a \neq c_b$.

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