How to Estimate the Stiffness of a Multi-Layer Road Based on Properties of Layers: Symmetry-Based Explanation for Odemark's Equation

Edgar Daniel Rodriguez Velasquez[1,2], Vladik Kreinovich[2], Olga Kosheleva[2], and Hoang Phuong Nguyen[3]

[1]Department of Civil Engineering
Universidad de Piura in Peru (UDEP), Piura, Peru
edgar.rodriguez@udep.pe
[2]University of Texas at El Paso, El Paso, TX 79968, USA,
edrodriguezvelasquez@miners.utep.edu
vladik@utep.edu, olgak@utep.edu
[3]Division Informatics, Math-Informatics Faculty
Thang Long University, Nghiem Xuan Yem Road
Hoang Mai District, Hanoi, Vietnam, nhphuong2008@gmail.com

1. Need to Estimate Stiffness of Multi-Layer Roads

- Most roads consist of several layers:
  - First, there is a layer of soil – if needed, stabilized by adding lime, cement, etc.
  - Then there is a layer – usually compacted – of crushed rocks.
  - Finally, an asphalt or concrete layer is placed on top.
- The road has to have a certain stiffness, i.e., a certain value of the modulus characterizing this stiffness.
- It is therefore desirable to estimate the stiffness of the designed road with the layers of given thickness:
  - we know the modulus E_i and the thickness h_i of each layer;
  - based on this information, we need to estimate the overall modulus E of the road.

2. Odemark's Equation

- A method for solving this problem was proposed in 1949 by N. Odemark: $E = \left(\dfrac{\sum\limits_i h_i \cdot \sqrt[3]{E_i}}{\sum\limits_i h_i}\right)^3.$
- This formula is still in use.
- This formula is based on a simplified mechanical model of the pavement.
- Here, many important factors are ignored to make a simple formula possible.
- In principle, several different simplifications are possible.
- The formula produced by this particular simplification has been confirmed by empirical data.
- How can we explain this formula?

3. What We Do in This Talk

- In this talk, we provide a theoretical explanation for this formula.
- This explanation is based on the ideas of symmetry – namely, on the ideas of scale-invariance.

4. Scale-Invariance: Reminder

- To measure a physical quantity, we need to select a measuring unit.
- In some cases, there is a physically natural unit.
- For example, in the micro-world, we can use the electric charge of an electron as a natural measuring unit for electric charges.
- However, in many other situations, there is no such fixed unit.
- In such cases, it is reasonable to require that:
  - the dependence between the physical properties remains the same,
  - i.e., is described by the same formula,
  - if we change the measuring unit.

5. Scale-Invariance (cont-d)

- If:
  - we replace the original measuring unit with a unit which is λ times smaller,
  - then all numerical values of the quantity will be multiplied by λ: $x \to \lambda \cdot x$.
- This transformation is known as *re-scaling*.
- Invariance with respect to this transformation is known as *scale-invariance*.
- Scale invariance is ubiquitous in physics.

6. Towards an Explanation

- Let us first consider the simplified case when all the layers have the same thickness.
- The overall stiffness E is the "average" stiffness.
- In other words, it is the stiffness that the road would have if all its layers have the same stiffness E.
- Let us denote the overall effect of n layers with stiffness $E_1, \ldots, E_n$ by $E_1 * \ldots * E_n$.
- Here, $a * b$ is an appropriate combination operation.
- In these terms, the stiffness of the n-layer road in which each layer has stiffness E is described by the formula

$$E * \ldots * E.$$

- Thus, the desired overall effect E can be described by the formula $E * \ldots * E = E_1 * \ldots * E_n$.

7. Towards an Explanation (cont-d)

- The air layer with 0 stiffness does not contribute to the overall stiffness, so we should have $a * 0 = a$.
- If we have layers of different thickness h_i, then we can:
  - divide each of these layers into parts of the same thickness, and
  - apply the same formula, i.e., we get

$$E * \ldots * E \ (h_1 + \ldots + h_n \text{ times}) =$$

$$E_1 * \ldots * E_1 \ (h_1 \text{ times}) * \ldots * E_n * \ldots * E_n \ (h_n \text{ times}).$$

8. Natural Properties of the Combination Operation $a * b$

- In the first approximation:
  - we can ignore the dependence on the order, and assume that $a * b = b * a$,
  - i.e., assume that the combination operation is commutative.
- It is also reasonable to assume that:
  - the result of applying this operation to a 3-layer road
  - does not depend on which layer we start with,
  - i.e., that we should have $a * b * c = (a * b) * c = a * (b * c)$.
- In other words, the combination operation should be associative.

9. Natural Properties of $a * b$ (cont-d)

- If we made one the layers stiffer, the stiffness of the road should increase.
- So, the combination operation should be strictly monotonic: if $a < a'$, then $a * b < a' * b$.
- Small changes in E_i should lead to small changes in the overall stiffness.
- In mathematical terms, this means that the combination operation should be continuous.
- Finally, we require that the combination operation be scale-invariant, i.e., that:
  - if $a * b = c$,
  - then, for every λ, we should have the same relation for re-scaled values $\lambda \cdot a$, $\lambda \cdot b$, and $\lambda \cdot c$:

$$(\lambda \cdot a) * (\lambda \cdot b) = \lambda \cdot c.$$

10. Main Result

- We will show that:
  - every commutative, associative, strictly monotonic, continuous, and scale-invariant combination operation for which $a * 0 = a$
  - has the form $a * b = (a^p + b^p)^{1/p}$ for some $p > 0$.
- In other words, $a * b = c$ is equivalent to $a^p + b^p = c^p$.
- In general, $a * \ldots * b = c$ means that $a^p + \ldots + b^p = c^p$.
- In our case, this means that $\left(\sum_i h_i\right) \cdot E^p = \sum_i h_i \cdot E_i^p$,

  hence $E = \left(\dfrac{\sum_i h_i \cdot E_i^p}{\sum_i h_i}\right)^{1/p}$.
- For $p = 1/3$, we get exactly Odemark's formula!

11. Proof: Part 1

- Let us first prove that the operation $a * b$ has the form $f(f^{-1}(a)+f^{-1}(b))$ for some monotonic function $f(a)$.
- In other words, we want to prove that $a * b = c$ is equivalent to $f^{-1}(a)+f^{-1}(b)=f^{-1}(c)$.
- Equivalently, we want to prove that that $f(a) * f(b) = f(c)$ is equivalent to $a+b=c$, i.e., that

$$f(a+b)=f(a)*f(b).$$

- Indeed, let us take $f(1) \stackrel{\text{def}}{=} 1$.
- Then, for every natural number m, we take

$$f(m) \stackrel{\text{def}}{=} 1 * \ldots * 1 \ (m \text{ times}).$$

- In this case indeed,

$$f(m)*f(m') = 1*\ldots*1 \ (m \text{ times})*1*\ldots*1 \ (m' \text{ times}) = 1 * \ldots * 1 \ (m+m' \text{ times}) = f(m+m').$$

12. Proof: Part 1 (cont-d)

- Due to monotonicity, for each natural number n:

$$0 * \ldots * 0 \ (n \text{ times}) = 0 < 0 * \ldots * 0 * 1 = 1, \text{ and}$$

$$1 = 0 * \ldots * 0 * 1 < 1 * \ldots * 1 \ (n \text{ times}).$$

- Here, $0 * \ldots * 0 \ (n \text{ times}) < 1 < 1 * \ldots * 1 \ (n \text{ times})$.
- The combination operation is continuous.
- Thus, there exists v_n for which

$$v_n * \ldots * v_n \ (n \text{ times}) = 1.$$

- We will then take $f\left(\frac{1}{n}\right) = v_n$.

13. Proof: Part 1 (cont-d)

- We will then define
$$f\left(\frac{m}{n}\right) \stackrel{\text{def}}{=} f\left(\frac{1}{n}\right) * \ldots * f\left(\frac{1}{n}\right) \quad (m \text{ times}).$$
- One can check that for this $f(a)$, we have $f(a+b) = f(a) * f(b)$ for rational a and b.
- By continuity, we can extend the function $f(a)$ to all non-negative real values a.

14. Proof: Part 2

- Let us now prove that the inverse function $f^{-1}(a)$ is a power function.
- Thus, its inverse is also a power function.
- Indeed, we have proven that $a * b = f(f^{-1}(a) + f^{-1}(b))$.
- Scale-invariance means that in this case, we have

$$f^{-1}(\lambda \cdot a) + f^{-1}(\lambda \cdot b) = f^{-1}(\lambda \cdot c).$$

- Let us denote $p \stackrel{\text{def}}{=} f^{-1}(a)$, $q \stackrel{\text{def}}{=} f^{-1}(b)$, $r \stackrel{\text{def}}{=} f^{-1}(c)$.
- Then, $a = f(p)$, $b = f(q)$, and $c = f(r)$.
- Let us also denote $t_\lambda(x) \stackrel{\text{def}}{=} f^{-1}(\lambda \cdot f(x))$, so that

$$t_\lambda(p) = f^{-1}(\lambda \cdot f(p)) = f^{-1}(\lambda \cdot a),$$
$$t_\lambda(q) = f^{-1}(\lambda \cdot f(q)) = f^{-1}(\lambda \cdot b). \text{ and}$$
$$t_\lambda(r) = f^{-1}(\lambda \cdot f(r)) = f^{-1}(\lambda \cdot c).$$

15. Proof: Part 2 (cont-d)

- So, scale-invariance takes the following form:
  - if $p + q = r$,
  - then $t_\lambda(p) + t_\lambda(q) = t_\lambda(r)$.
- In other words, we have $t_\lambda(p+q) = t_\lambda(p) + t_\lambda(q)$ for all p and q.
- For integer values $p = n$, we thus have

$$t_\lambda(1) = t_\lambda\left(\frac{1}{n}\right) + \ldots + t_\lambda\left(\frac{1}{n}\right) \ (n \text{ times}) = n \cdot t_\lambda\left(\frac{1}{n}\right).$$

- Thus $t_\lambda\left(\frac{1}{n}\right) = \frac{1}{n} \cdot t_\lambda(1)$.

16. Proof: Part 2 (cont-d)

- Similarly, for every m, we have

$$t_\lambda\left(\frac{m}{n}\right) = t_\lambda\left(\frac{1}{n}\right) + \ldots + t_\lambda\left(\frac{1}{n}\right) \ (m \text{ times}) =$$

$$m \cdot t_\lambda\left(\frac{1}{n}\right) = \frac{m}{n} \cdot t_\lambda(1).$$

- In other words, we conclude that $t_\lambda(x) = x \cdot t_\lambda(1)$ for all rational x.
- By continuity, we can conclude that this property holds for all real values as well.
- By definition of $t_\lambda(x)$, the equality $t_\lambda(x) = x \cdot t_\lambda(1)$ means that $f^{-1}(\lambda \cdot f(x)) = t_\lambda(1) \cdot x$.
- In other words, for the value $y = f(x)$, for which $x = f^{-1}(y)$, we have $f^{-1}(\lambda \cdot y) = t_\lambda(1) \cdot f^{-1}(y)$.

17. Proof: Part 2 (cont-d)

- It is known that every continuous solution to this functional equation has the form

$$f^{-1}(x) = A \cdot x^a \text{ for some } A \text{ and } a.$$

- Thus, we get the desired formula for the combination operation $a * b = f(f^{-1}(a) + f^{-1}(b))$.
- The result is proven.

18. Comment

- The functional equation result can be easily proven, if:
  - instead of continuity,
  - we make a stronger assumption that the combination operation is differentiable, and
  - thus, the function $f(a)$ is differentiable.
- Indeed, in this case, $t_\lambda(1)$ is a differentiable function of λ, as a ratio of two differentiable functions.
- Thus, we can differentiate both sides of the equality $f^{-1}(\lambda \cdot y) = t_\lambda(1) \cdot f^{-1}(y)$ by λ and take $\lambda = 1$.
- Then, we get $x \cdot F'(x) = c \cdot F$, where:
  - $F(x) \stackrel{\text{def}}{=} f^{-1}(x)$,
  - $F'(x)$ means the derivative, and
  - c is the derivative of $t_\lambda(1)$ when $\lambda = 1$.

19. Comment (cont-d)

- This formula can be rewritten as $x \cdot \dfrac{dF}{dx} = c \cdot F$, i.e., equivalently, $\dfrac{dF}{F} = c \cdot \dfrac{dx}{x}$.
- Integrating both parts, we get $\ln(F) = c \cdot \ln(x) + C$, where C is the integration constant.
- Applying $\exp(z)$ to both sides, we get the desired power law.

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