Why Some Power Laws Are Possible And Some Are Not

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1. Power Laws Are Ubiquitous

- In many application areas, the dependence between two quantities x and y is described by the formula

$$y = A \cdot x^a \text{ for some } a \text{ and } A.$$

- Such dependencies are known as *power laws*.
- Power laws are truly ubiquitous.
- They describe how the aerodynamic resistance force depends on the plane's velocity.
- They describe how the perceived signal depends on the intensity of the signal that we hear and see.
- They describe how the mass of celestial structures depends on the structure's radius, etc.

2. Sometimes, Not All Power Laws Are Possible

- The parameters A and a have to be determined from the experiment.
- In some application areas, all pairs (A, a) are possible.
- In some other applications areas, however, not all such pairs are possible.
- Sometimes, a is fixed, and A can take all possible values.
- In other application areas:
  - we have different values of A,
  - but for each A, we can only one have one specific value of a.

3. Not All Power Laws Are Possible (cont-d)

- One such example can be found in transportation engineering.
- It describes the dependence of number y of cycles until fatigue failure on the initial strain x.
- In many such situations, the value of a corresponding to A is determined by the following empirical formula

$$a = c_0 + c_1 \cdot \ln(A).$$

- The case when the value a is fixed can be viewed as a particular case $c_1 = 0$ of this empirical formula.

4. Resulting Challenge

- How can we explain the formula $a = c_0 + c_1 \cdot \ln(A)$?
- In this talk, we provide a theoretical explanation for this formula.
- To come up with this explanation:
  - we recall the reason why power laws are ubiquitous in the first place
  - because they correspond to scale-invariant dependencies.
- We then use the scale-invariance idea to explain the ubiquity of the desired formula.

5. Power Laws and Scale Invariance

- The main purpose of data processing is to deal with physical quantities.
- However, in practice, we only deal with the numerical values of these quantities.
- What is the difference?
- The difference is that:
  - to get a numerical value,
  - we need to select a measuring unit for measuring the quantity.
- If:
  - we replace the original measuring unit with a new one which is λ times smaller,
  - then all numerical values are multiplied by λ:

$$x \to X = \lambda \cdot x.$$

6. Power Laws and Scale Invariance (cont-d)

- For example, if we move from meters to centimeters:
  - all the numerical values will be *re-scaled*: multiplied by 100;
  - e.g., 1.7 m becomes $1.7 \cdot 100 = 170$ cm.

7. Scale-Invariance

- In many application areas, there is no fixed measuring unit.
- The choice of the measuring unit is rather arbitrary.
- In such situations, it is reasonable to require that:
  - the dependence $y = f(x)$ between the quantities x and y
  - should not depend on the choice of the unit.
- Of course, this does not mean that $y = f(x)$ imply $y = f(X) = f(\lambda \cdot x)$ for the exact same function $f(x)$.
- That would mean that $f(\lambda \cdot x) = f(x)$ for all x and λ.
- So $f(x)$ is a constant and thus, that there is no dependence.

8. Scale-Invariance (cont-d)

- What we need to do to keep the same dependence is:
  - to accordingly re-scale y,
  - to $Y = \mu \cdot y$ for some μ depending on λ.
- For example, the area y of a square is equal to the square of its size $y = x^2$.
- This formula is true if we use meters to measure length and square meters to measure area.
- The same formula holds if we use centimeters instead of meters.
- However, then, we should use square centimeters instead of square meters.
- In this case, $\lambda = 100$ corresponds to $\mu = 10000$.

9. Scale-Invariance (cont-d)

- So, we arrive at the following definition of scale-invariance:
  - for every $\lambda > 0$ there exists a value $\mu > 0$ for which, for every x and y,
  - the relation $y = f(x)$ implies that $Y = f(X)$ for $X = \lambda \cdot x$ and $Y = \mu \cdot y$.

10. Scale-Invariance and Power Laws

- It is easy to check that every power law is scale-invariant.
- Indeed, it is sufficient to take $\mu = \lambda^a$.
- Then, from $y = A \cdot x^a$ we get

  $$Y = \mu \cdot y = \lambda^a \cdot y = \lambda^a \cdot A \cdot x^a = a \cdot (\lambda \cdot x)^a = a \cdot X^a.$$
- So, indeed $Y = f(X)$.
- It turns out that, vice versa, the only continuous scale-invariance dependencies are power laws.
- For differentiable functions $f(x)$, this can be easily proven.
- Indeed, by definition, scale-invariance means that $\mu(\lambda) \cdot f(x) = f(\lambda \cdot x)$.
- Since $f(x)$ is differentiable, $\mu(\lambda) = \dfrac{f(\lambda \cdot x)}{f(x)}$ is also differentiable, as the ratio of two differentiable functions.

11. Scale-Invariance and Power Laws (cont-d)

- Since $f(x)$ and $\mu(\lambda)$ are differentiable, we can differentiate the equality $\mu(\lambda)\cdot f(x) = f(\lambda\cdot x)$ w.r.t. λ:

$$\mu'(\lambda)\cdot f(x) = x\cdot f'(\lambda\cdot x).$$

- In particular, for $\lambda = 1$, we get $\mu_0\cdot f(x) = x\cdot f'(x)$, where $\mu_0 \stackrel{\text{def}}{=} \mu'(1)$, so $\mu_0\cdot f = x\cdot \dfrac{df}{dx}$.
- We can separate the x and f is we divide both sides by $x\cdot f$ and multiply by dx: $\dfrac{df}{f} = \mu_0\cdot\dfrac{dx}{x}$.
- Integrating both sides, we get $\ln(f) = \mu_0\cdot\ln(x) + c$, where c is the integration constant.
- Thus, for $f = \exp(\ln(f))$, we get

$$f(x) = \exp(\mu_0\cdot\ln(x) + c) = A\cdot x^a.$$

- Here $A \stackrel{\text{def}}{=} \exp(c)$ and $a \stackrel{\text{def}}{=} \mu_0$.

12. Main Idea

- In principle, for the corresponding application areas, we can have different values A and a.
- This means that the value of the quantity y is not uniquely determined by the value of the quantity x.
- There must be some other quantity z that influences y: $y = F(x, z)$.
- Different situations – i.e., different pairs (A, a) – are characterized by different values of the quantity z.

13. Main Assumption

- For each fixed z, the dependence of y on x is described by a power law.
- Thus, when the value of z is fixed, the dependence of y on x is scale-invariant.
- It is therefore reasonable to conclude that, vice versa:
  - for each fixed value x,
  - the dependence of y on z is also scale-invariant.

14. This Assumption Leads to the Desired Explanation

- Let us show that this assumption indeed explains the desired formula.
- For each z, the dependence of y on x is a power law:

$$F(x, z) = A(z) \cdot x^{a(z)}.$$

- Similarly, for each x, the dependence of y on z is also a power law: $F(x, z) = B(x) \cdot z^{b(x)}$.
- Thus, $A(z) \cdot x^{a(z)} = B(x) \cdot z^{b(x)}$ for all x and z.

15. Explanation (cont-d)

- In particular, for $x = 1$, we get $A(z) = B(1) \cdot z^{b(1)}$.
- Similarly, for $z = 1$, we get $B(x) = A(1) \cdot x^{a(1)}$.
- Substituting these expressions into the above equality, we get $B(1) \cdot z^{b(1)} \cdot x^{a(z)} = A(1) \cdot x^{a(1)} \cdot z^{b(x)}$.
- In particular, for $x = e$, we get
$$B(1) \cdot z^{b(1)} \cdot e^{a(z)} = A(1) \cdot e^{a(1)} \cdot z^{b(e)}.$$
- So, $\exp(a(z)) = \dfrac{A(1) \cdot \exp(a(1))}{B(1)} \cdot z^{b(e)-b(1)}$.
- From $A(z) = B(1) \cdot z^{b(1)}$, we conclude that $z^{b(1)} = \dfrac{A}{B(1)}$.
- Thus, $z = \dfrac{A^{1/b(1)}}{B(1)^{1/b(1)}}$.

16. Explanation (cont-d)

- *Reminder:* $z = \dfrac{A^{1/b(1)}}{B(1)^{1/b(1)}}$.
- Substituting this expression for z into the formula for $\exp(a)$, we get:

$$\exp(a) = \frac{A(1) \cdot \exp(a(1))}{B(1)} \cdot \frac{1}{B(1)^{(b(e)-b(1))/b(1)}} \cdot A^{(b(e)-b(1))/b(1)}.$$

- So, $\exp(a) = C_0 \cdot A^{c_1}$ for some values C_0 and c_1.
- Taking logarithms of both sides, we now get the desired dependence $a = c_0 + c_1 \cdot \ln(A)$, where $c_0 \stackrel{\text{def}}{=} \ln(C_0)$.
- So, we indeed have the desired derivation.

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