How the Pavement's Lifetime Depends on the Stress Level and on the Dry Density: An Explanation of Empirical Formulas

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1. Formulation of the first problem

- Road pavements have a limited lifetime.
- As the vehicles pass over the pavement, eventually, the pavement develops fatigue cracking and needs to be repaired.
- Clearly, the larger the load, the larger the stress σ, the larger the strain ε.
- Thus, the larger the load, the smaller the number of repetitions N before the fatigue cracking.
- To estimate the expected lifetime of the pavement, we need to know how:
  - the number of repetitions N to fatigue cracking depends
  - on the stress σ.
- So, we need to know the dependence $N = f(\sigma)$.

2. Formulation of the first problem (cont-d)

- Empirically, this dependence is described by the following formula:

$$N = N_0 \cdot \exp(-k \cdot \sigma).$$

- This formula applies both to the cases when we analyze how N depends:
  - on the stress measured at the top layer of the pavement and
  - on the stress measured at the bottom layer of the pavement.
- How can we explain this empirical formula?

3. There are several components of stress

- Usually, we implicitly assume that all this stress is caused by the traffic.
- In reality, in addition to the traffic-related stress, there are other stresses that also contribute to the deterioration of the pavement.
- Indeed, even an unused road eventually develops cracks, due, e.g., to weather-induced stress.
- The additional stress – e.g., weather-related stress – follows different cyclic patterns than the traffic-related stress.
- Typically, it follows a yearly pattern, or – in case of the rain – a pattern corresponding to each instance of rain.
- The number of repetitions of these additional stresses is therefore much smaller than the number of repetitions caused by the traffic.
- Thus, in comparison with the number of traffic-related repetitions, the number of weather-related repetitions can be safely ignored.

4. There are several components of stress (cont-d)

- Strictly speaking, only the traffic-related part σ_t of the stress affects the number of repetitions N.
- In other words, we should look for a formula $N = f(\sigma_t)$.

5. How can we estimate the traffic-related part of the stress

- All we measure is the overall stress σ, which is equal to the sum $\sigma_t + \sigma_o$ of the traffic-related stress σ_t and the stress σ_o caused by other factors.
- So, to estimate the traffic-related stress σ_t, we need to subtract, from the measured stress σ, our estimate of the remaining stress σ_o.
- The problem is that this remaining stress can only be estimated rather approximately:
  - if instead of the original estimate σ_o we used a slightly different estimate σ'_o,
  - then, instead of the original estimate $\sigma_t = \sigma - \sigma_o$, we get a somewhat different estimate

$$\sigma'_t = \sigma - \sigma'_o = (\sigma - \sigma_o) + (\sigma_o - \sigma'_o) = \sigma_t + \delta, \text{ where } \delta \stackrel{\text{def}}{=} \sigma_o - \sigma'_o.$$

- As a result, the exact same situation can be described by two somewhat different values σ_t and $\sigma'_t = \sigma_t + \delta$.

6. Natural invariance idea

- The exact same situation can be described by two somewhat different values σ_t and $\sigma_t = \sigma'_t + \delta$.
- It is therefore reasonable to require that these two somewhat different values lead to the exact same predictions of the pavement lifetime.
- Of course, we cannot interpret this requitement literally, as saying that we should have $f(\sigma_t) = f(\sigma_t + \delta)$:
  - if we impose this requirement for all σ_t and all δ,
  - this would lead to a physically meaningless conclusion that the function $f(\sigma_t)$ is a constant, i.e.,
  - that the pavement's lifetime does not depend on the stress at all.

7. Solution: using experience of physics

- The experience of physics shows that we do not need to take this requirement literally.
- For example, we know that:
  - the formula $v = d/t$ describing the velocity v as a function of distance d and time t
  - does not depend on what measuring unit we select for distance.
- We can describe the distance d in meters, or we can describe it in centimeters, resulting in a different numerical value $d' = 100 \cdot d$.
- In this example, invariance does not mean $d/t = d'/t$.
- The formula $v = d/t$ does remain valid if we use a different unit for measuring distance.
- However, for this formula to remain valid, we also need to correspondingly change the unit that we use for measuring velocity.
- In the above example, from m/sec to cm/sec.

8. Solution: using experience of physics (cont-d)

- In this case, we get $v' = d'/t$, where v' is the numeric value of velocity in the new units.
- From this viewpoint, a reasonable idea is to require that for $\sigma'_t = \sigma_t + \delta$:
  - the dependence $N = f(\sigma_t)$ should lead to $N' = f(\sigma'_t)$,
  - where N' is the description of the pavement lifetime in correspondingly different units.

9. But can we use the experience of physics in our case?

- At first glance, the physics-motivated idea does not seem to work in our case:
  - in contrast to quantities like distance or velocity, where we do need to select a measuring unit to get a numerical value,
  - the number of repetitions N is simply an integer, no measuring is required.
- However, a more detailed analysis shows that the situation is not that uniquely determined.
- Indeed, usually, when the vehicle goes over a pavement location:
  - we can count it as a single stress cycle,
  - or we can take into account that every time each wheel is passing over, it is a different cycle.
- Thus, depending on how we count it, what was a single cycle in one counting becomes several cycles if we consider it differently.

10. Can we use the experience of physics in our case (cont-d)

- In mathematical terms, this means that we can describe the same traffic history:
  - by the number N and
  - by a different number $N' = c \cdot N$, where c is the average number of axles per vehicle.
- Different estimates of the number c can lead, in general, to different re-scalings of N'.

11. Resulting formulation of the invariance requirement

- The above requirement that the dependence $N = f(\sigma_t)$ not change if we change our estimate for σ_o takes the following form:
  - for every real number δ,
  - there exists an appropriate value $c(\delta)$ – which depends on δ,
  - such that if $N = f(\delta_t)$, then we should have $N' = f(\sigma'_t)$, where $\sigma'_t = \sigma_t + \delta$ and $N' = c(\delta) \cdot N$.

12. Reduction to a functional equation

- Substituting the expressions $\sigma'_t = \sigma_t + \delta$ and $N' = c(\delta) \cdot N$ into the formula $N' = f(\sigma'_t)$, we conclude that $c(\delta) \cdot N = f(\sigma_t + \delta)$.
- Since $N = f(\sigma_t)$, we thus conclude that

$$c(\delta) \cdot f(\sigma_t) = f(\sigma_t + \delta).$$

- It is known that every measurable solution to this functional equation has the form $f(\sigma) = N_0 \cdot \exp(-k \cdot \sigma)$.
- Thus, we have indeed explained the above empirical formula.

13. Second problem: dependence on dry density

- The above formula describes how the pavement lifetime depends on the stress provided that all other parameters remain constant.
- The corresponding values N and k depend on the dry density ρ of the underlying soil.
- This dry density is called *maximum dry density* since when the pavement is built, we try to maximize the dry density of the compacted soil.
- The dependence of the lifetime N on stress and dry density is described by the following formula:

$$\ln(N) = k_4 \cdot \ln\left(\frac{\rho}{\omega}\right) \cdot \left(1 - \frac{\sigma}{k_5 \cdot UCS}\right) \text{ for some } k_4, \omega, k_5, UCS.$$

- This is equivalent to:

$$\ln(N) = a_{00} + a_{01} \cdot \sigma + a_{10} \cdot \ln(\rho) + a_{11} \cdot \sigma \cdot \ln(\rho), \text{ for some } a_{00}, a_{01}, a_{10}, a_{11}.$$

- How can we explain this empirical formula?

14. What we know already

- We know, from the previous text, that for each fixed value ρ, $\ln(N)$ is a linear function of σ:

$$\ln(N) = \ln(N_0) - k \cdot \sigma.$$

- Let us analyze how N depends on the dry density ρ.

15. Corresponding invariance

- We can use different units to measure dry density.
- If we replace the original unit with a new unit which is λ times smaller, then all numerical values get multiplied by λ.
- Instead of each original numerical value ρ, we get a new numerical value $\rho' = \lambda \cdot \rho$ to describe the exact same physical situation.
- There is no reason to prefer one specific measuring unit.
- It is reasonable to require that the dependence of the lifetime N on the density ρ be the same, no matter what measuring unit we use.
- It is reasonable to formulate this requirement in the following terms:
  - for every real number $\lambda > 0$, there exists an appropriate value $c(\lambda)$ – which depends on λ
  - such that if $N = f(\rho)$, then we should have $N' = f(\rho')$, where $\rho' = \lambda \cdot \rho$ and $N' = c(\lambda) \cdot N$.

16. Reduction to a functional equation

- Substituting the expressions $\rho' = \lambda \cdot \rho$ and $N' = c(\lambda) \cdot N$ into the formula $N' = f(\rho')$, we conclude that $c(\lambda) \cdot N = f(\lambda \cdot \rho)$.
- Since $N = f(\rho)$, we thus conclude that $c(\lambda) \cdot f(\rho) = f(\lambda \cdot \rho)$.
- It is known that every measurable solution to this functional equation has the form $f(\rho) = N_0 \cdot \rho^a$ for some values N_0 and a.
- Thus, $\ln(N) = a \cdot \ln(\rho) + \ln(N_0)$, i.e., $\ln(N)$ is a linear f-n of ρ; so:
  - for each value ρ, the logarithm $\ln(N)$ of the lifetime $N = f(\sigma, \rho)$ is a linear function of σ, and
  - for each value σ, the logarithm $\ln(N)$ is a linear function of $\ln(\rho)$.
- Thus, the function $\ln(N)$ is a bilinear function of σ and $\ln(\rho)$, i.e., has the desired form.
- So, the empirical dependence on ρ is also explained.

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