How to Work? How to Study? Shall We Cram for the Exams? And How Is This Related to Life on Earth?

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1. Need to switch activities

- People get tired when doing the same work for a long time, or studying the same material for a long time.
- As time goes, their productivity decreases.
- The best way to restore productivity is to switch to a different activity – or to some relaxation – and then get back to the original activity.
- On the other hand, too many switches decrease productivity as well.
- Indeed, a person needs some time to become productive when switching to a new activity.
- There are many examples of such a decrease in productivity.
- For example, constant interruptions – like immediate replies to emails and/or to phone calls – decrease productivity.

2. Need to switch activities (cont-d)

- Historically, this was one of the reasons why switching from a 6-day work week to a 5-day work week increased productivity.
- Crudely speaking, the first hour of each work day is not very productive, so the fewer such unproductive hours per week, the better.

3. This effect drastically varies from one person to another

- This effect is different for different people.
- Some students cram for the exam by studying for many hours in a row – and do well.
- Other students try cramming and fail.
- During a 2-hour-long class:
  - some students urge the instructor for a break after the first hour, since their ability to understand decreases,
  - while others urge to continue, since they do not want to lose the track.
- Some workers prefer to work through the lunch break and go home earlier.
- Others need the whole lunch break to restore their productivity.

4. Problem

- It takes some time for people to find their best switching schedule.
- During this time, their productivity is not the best.
- They may be switching too rarely getting less productive at the end of each work spurt.
- They may be switching too frequently, wasting too much time on switching.
- It is therefore desirable to help people by providing individualized recommendations on how to switch.
- Coming up with such recommendations is the main objective of this paper.

5. How we get tired

- As we start performing some activity, after a short period of adjustment, we reach a reasonable productivity level p_0.
- This is the day's maximum productivity level.
- Let us take the moment of time when we reach this productivity level as the starting point $t = 0$ for measuring time.
- So, the productivity $p(t)$ at moment $t = 0$ is equal to p_0: $p(0) = p_0$.
- As we continue performing the same activity, our productivity $p(t)$ decreases, so its derivative $\dot{p}(t)$ is negative.
- How can we describe this decrease?
- The rate $\dot{p}(t)$ at which productivity decreases, in general, depends on the original productivity level: $\dot{p}(t) = f(p(t))$ for some function $f(p)$.
- We are not considering extreme cases, when a person works at the limit of his/her abilities.

6. How we get tired (cont-d)

- These situations are rare, since it is not possible to maintain such an extreme productivity all the time.
- Usually, our productivity is much smaller that this maximum amount.
- The usual productivity p is reasonably small.
- So, we can expand the dependence $f(p)$ in Taylor series and keep only the few first terms in this expansion.
- In particular, if we only keep linear terms, we conclude that $f(p) = a_0 + a_1 \cdot p$ for some constants a and b.
- When the person is so tired that his/her productivity is close to 0, this productivity will stay at close to 0.
- Indeed, there is no room for any further decrease.
- So, we have $f(0) = 0$, which implies that $a_0 = 0$ and thus, $f(p) = a_1 \cdot p$.
- Since productivity decreases, we have $f(p) < 0$, i.e., $a_1 < 0$.

7. How we get tired (cont-d)

- Thus, $f(p)=-q\cdot p$, where we denoted $q\stackrel{\text{def}}{=}|a_1|$.
- From the equation $\dot p(t)=-q\cdot p(t)$, taking into account that $p(0)=p_0$, we conclude that $p(t)=p_0\cdot \exp(-q\cdot t)$.
- This formula is similar to the usual decay formulas – e.g., to the formulas describing the radioactive decay.
- The rate of radioactive decay is usually described by *half-life*, the time h at which we are left with the half of the original amount.
- Similarly, let us gauge our rate of becoming tired by the time h at which our productivity decreases to $p_0/2$.
- This time is related to the value q by the formula $p_0\cdot\exp(-q\cdot h)=p_0/2$, i.e., $\exp(-q\cdot h)=1/2$ and thus, $q=\dfrac{\ln(2)}{h}$.

8. How we recover

- Once we switch to a new activity, we need some time to gain the optimal productivity.
- Let us denote the switch-caused lost time by t_0.

9. Formulation of the problem

- Suppose we plan an activity for which we allocated time T.
- If we perform it without taking a break, then the overall productivity P during this time can be obtained by integrating $p(t)$:

$$P = \int_0^T p_0 \cdot \exp(-q \cdot t)\, dt = p_0 \cdot \left. \frac{\exp(-q \cdot t)}{-q} \right|_0^T = p_0 \cdot \frac{1 - \exp(-q \cdot T)}{q}.$$

- On the other hand, if we take a break after time T_1, then we lose time t_0 on adjustment, and continue working for time $T - t_0 - T_1$.
- Our overall productivity us then the sum of the productivities during these two periods of time, and is, thus, equal to

$$p_0 \cdot \frac{1 - \exp(-q \cdot T_1)}{q} + p_0 \cdot \frac{1 - \exp(-q \cdot (T - t_0 - T_1))}{q}.$$

10. Formulation of the problem (cont-d)

- First natural question: when is it beneficial to take a break?
- Clearly, it is not beneficial if the time T is short, and it is beneficial if T is long.
- What is the threshold value T_0 starting from which the break will be beneficial?
- If it is beneficial to take a break, when should we take it?
- What is the value T_1 that leads to the largest overall productivity?

11. If a break, when?

- Let us first find the optimal value T_1.
- Possible values T_1 comes from the interval $[0, T - t_0]$.
- The duration T_2 of the second phase is $T_2 = T - t_0 - T_1$.
- According to calculus, the optimal value of T_1 is attained:
  - either at one of the endpoints, when $T_1 = 0$ or $T_2 = 0$,
  - or inside the interval, when the derivative of the productivity with respect to T_1 is equal to 0.
- Equating the derivative of the objective function to 0, we get

$$p_0 \cdot exp(-q \cdot T_1) - p_0 \cdot \exp(-q \cdot (T - t_0 - T_1)) = 0.$$

- This implies that $T_1 = T - t_0 - T_1$ and thus, that $T_1 = T_2 = \frac{T - t_0}{2}$.

12. If a break, when (cont-d)

- The productivity corresponding to $T_1 = 0$ or $T_2 = 0$ is smaller.
- Indeed, for the first half of the interval of length $T - t_0$, it coincides with what we have for $T_1 = T_2$, and after that:
  - in the $T_1 = T_2$ case, we start afresh, with productivity p_0,
  - while in the $T_i = 0$ cases, we start with a tired state.
- So, the optimal value T_1 is inside the interval, when $T_1 = T_2$.
- Thus, if we need a break, we need to make it right in the middle of the activity.
- In this case, the overall productivity is equal to

$$2 \cdot p_0 \cdot \frac{1 - \exp(-q \cdot (T/2 - t_0/2))}{q}.$$

13. What if we need several breaks?

- Let's consider the case when we schedule B breaks.
- Similarly, we can show that the maximal productivity is attained when the corresponding work time intervals $T_1, \ldots, T_{B+1}$ are equal:

$$T_1 = \ldots = T_{B+1} = \frac{T - B \cdot t_0}{B+1}.$$

- In this case, the overall productivity is equal to

$$(B+1) \cdot p_0 \cdot \frac{1 - \exp(-q \cdot (T/(B+1) - B \cdot t_0/(B+1)))}{q}.$$

14. How many breaks do we need?

- The overall time of breaks $B \cdot t_0$ cannot exceed the allocated time T.
- So we only need to consider values B for which $B \cdot t_0 < T$, i.e., values $B < T/t_0$.
- To achieve the maximal productivity, we need to select the value $B = 0, 1, 2, \ldots, \lfloor T/t_0 \rfloor$ for which the productivity is the largest.
- All these expressions (6) are proportional to p_0 and inverse proportional to q.
- So to decide which one if larger, it is sufficient to compare coefficients at p_0/q at these expressions, i.e., the values

$$(B+1) \cdot (1 - \exp(-q \cdot (T/(B+1) - B \cdot t_0/(B+1))).$$

15. Do we need a break at all?

- To decide whether we need a break, we need to compare the values corresponding to $B = 0$ (no breaks) and $B = 1$ (one break).
- We need a break if the value corresponding to $B = 1$ is larger, i.e., if

$$2 \cdot (1 - \exp(-q \cdot (T/2 - t_0/2))) > 1 - \exp(-q \cdot T).$$

- If we denote $z \stackrel{\text{def}}{=} \exp(-q \cdot (T/2))$, then this inequality takes the form $2 - 2\alpha \cdot z > 1 - z^2$, where we denoted $\alpha \stackrel{\text{def}}{=} \exp(q \cdot t_0/2)$.
- This inequality is equivalent to $z^2 - 2\alpha \cdot z + 1 > 0$.
- This inequality is satisfied if z is:
  - either smaller that the smaller α_- of the two roots of the corresponding quadratic equation $z^2 - 2\alpha \cdot z + 1 = 0$,
  - or larger than the larger toot α_+.
- The roots of this quadratic equation are equal to

$$\alpha_\pm = \alpha \pm \sqrt{\alpha^2 - 1}.$$

16. Do we need a break at all (cont-d)

- Here, $\alpha = \exp(q \cdot t_0/2) > 1$, so $\alpha_+ > 1$, but $z = \exp(-q \cdot T/2) < 1$, so we cannot have $z > \alpha_+$.
- Thus, the break is needed if z is smaller than the smaller of the two roots, i.e., if $\exp(-q \cdot T/2) < \alpha_- = \alpha - \sqrt{\alpha^2 - 1}$.
- The decrease in productivity during the break time t_0 is small, so $\exp(-q \cdot t_0) \approx 1$ and thus, the product $q \cdot t_0$ is small.
- Thus, we can safely consider only the first few terms in the Taylor expansions when analyzing this formula.
- Hence, $\alpha = \exp(q \cdot t_0/2) \approx 1 + q \cdot t_0/2$, $\alpha^2 - 1 = \exp(q \cdot t_0) - 1 \approx 1 + q \cdot t_0 - 1 = q \cdot t_0$, and thus, $\alpha_- = \alpha - \sqrt{\alpha^2 - 1} \approx 1 + q \cdot t_0/2 - \sqrt{q \cdot t_0}$.
- Since the product $q \cdot t_0$ is small, its square root is much larger than the value itself.
- So, in comparison with the square root, the term $q \cdot t_0/2$ can be safely ignored, and we get $\alpha_- = \alpha - \sqrt{\alpha^2 - 1} \approx 1 - \sqrt{q \cdot t_0}$.

17. Do we need a break at all (cont-d)

- So, we need a break if $\exp(-q\cdot T/2)<1-\sqrt{q\cdot t_0}$.
- Taking the logarithm of both sides and taking into account that for small $q\cdot t_0$, we get $\ln(1-\sqrt{q\cdot t_0})\approx -\sqrt{q\cdot t_0}$, we conclude that

$$-q\cdot T/2<-\sqrt{q\cdot t_0}.$$

- This is equivalent to $T>2\cdot\sqrt{\dfrac{t_0}{q}}$.
- Substituting $q=\ln(2)/h$, we get $T>\dfrac{2}{\ln(2)}\cdot\sqrt{t_0\cdot h}$.
- So, we arrive at the following recommendations.

18. Resulting recommendations

- Let h be the time during which a person's productivity drops to half of its original value.
- Let t_0 be the time needed to get to speed when switching to a new activity.
- Let T be the time allocated to a certain activity.
- We will denote $q \stackrel{\text{def}}{=} \ln(2)/h$.
- In general, the number of breaks B can be between 0 (no breaks) and the largest possible value T/t_0.
- The optimal number of breaks B_{opt} is attained when the productivity is the largest:

$$B_{\text{opt}} = \arg\max_B [(B+1) \cdot (1 - \exp(-q \cdot (T/(B+1) - B \cdot t_0/(B+1)))].$$

19. Resulting recommendations (cont-d)

- In particular, we need a break at all if the time T exceeds the following threshold value: $T_0 = \frac{2}{\ln(2)} \cdot \sqrt{t_0 \cdot h}$.
- Here, the ratio $2/\ln(2)$ is approximately equal to 3.

20. Examples

- Let us consider the case when the recovery time t_0 is 1 hour, and the half-life is $h = 4$ hours – half or the usual workday.
- Then we need a break when the overall time is larger than $3 \cdot \sqrt{1 \cdot 4} = 6$ hours.
- This explains why most people need a full lunch break during a usual 8-hours working day.
- In studying, the typical recovery time is $t_0 = 10$ minutes (typical interval between classes), and $h = 50$ minutes – a typical class time.
- Then we need a break when the class time is larger than $3 \cdot \sqrt{10 \cdot 50} \approx 70$ minutes.
- In effect, we need a break during each class which is longer than normal – definitely we need a break for a 2-hour class.

21. If we need breaks, when do we schedule them?

- Once we found the optimal number of breaks $B_{\text{opt}} > 0$, we need to divide the original task into $B+1$ smaller parts $T_1, \ldots, T_{B+1}$.
- The optimal productivity is when we divide the time $T - B \cdot t_0$ (that remains after subtracting the breaks time) into $B+1$ equal durations:

$$T_1 = \ldots = T_{B+1} = \frac{T - B \cdot t_0}{B+1}. \tag{16}$$

22. How is this related to life on Earth?

- In the previous sections, we talked about people needing time to get up to speed when switching to a new activity.
- However, this phenomenon is generic, it is typical to all the living creatures.
- In particular, it turned out that bacteria that produce oxygen need some time to switch to the most productive regime; ss a result:
  - when many years ago, the Earth was rotating faster and a day lasted only 6 hours,
  - a big proportion of that time was spent on adjusting.
- When the Earth's rotation slowed down to the current 24-hour day, this drastically increased the bacteria productivity.
- The resulting drastic increase in the amount of oxygen in the Earth's atmosphere led to a boost of other life forms; see reference.

23. Reference

J. M. Klatt, A. Chennu, B. K. Arbic, B. A. Biddanda, and G. J. Dick, "Possible link between Earth's rotation rate and oxygenation", *Nature Geoscience*, 2021, Vol. 14, pp. 564–570
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