How to Combine Expert Estimates?
How to Estimate Probability in the Intersection of Two Populations?

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1. The first problem

- Suppose that the experts $E_1$ and $E_2$ provided two estimates $p_1$ and $p_2$ for the probability of some event $E$.
- We would like to provide a single estimate that takes both estimates into account.
- To properly combine the two estimates, it is important to take into account how related are the opinions of the two experts.
- If can happen that in all previous situations and in this situation, the experts gave almost identical opinions.
- This probably means that they use the same technique to provide their estimates.
- In this case, the opinion of the second expert does not add anything new to the opinion of the first expert.
- So, the combined probability will still be the same value $p_1$. 
2. The first problem (cont-d)

- It may happen that in the previous situations, the experts’ opinions were independent.

- This means that they use different data and different techniques to estimate the probability.

- In this case, e.g.:
  - if both experts believe that this event is possible,
  - then taking both opinions into account should increase this probability.
3. The first problem (cont-d)

- On the other hand, if the expert opinions are negatively correlated, then we do not know whom to believe.
- Then, we should not take either of the probabilities seriously.
- In this case, the combined probability should be close to the do-not-know 0.5 value.
- In this talk, we show how to come up with a reasonable numerical value of the combined probability.
4. Formulation of the second problem

- Suppose that we know the frequencies of a certain phenomenon in two different populations.

- E.g., we know the frequency of a certain disease in a 50-60 age group and the frequency of this disease in women.

- What is the reasonable estimate for the frequency of this disease in the intersection of these two populations.

- E.g., among women in the 50-60 age bracket?

- In this paper, we show that this problem is mathematically similar to the first one.

- Thus, all the methods for solving the first problem can be automatically applied to the second problem as well.
5. Case of the first problem

- Let $E$ be equal to 1 if the event happens and 0 if it does not.
- Let $e_1$ be equal to 1 if the first expert is correct in a randomly selected situation, and 0 if the first expert is wrong.
- Similarly, let $e_2$ be equal to 1 if the second expert is correct in a randomly selected situation, and 0 if the second expert is wrong.
- We know conditional probabilities $p(E | e_1) = p_1$, $p(E | e_2) = p_2$.
- Based on the analysis of the previous expert opinions, we can estimate the probability $p(e_1)$.
- We can do this by counting how many times the first expert was right.
- Similarly, based on the analysis of the previous expert opinions, we can estimate the probability $p(e_2)$. 
6. Case of the first problem (cont-d)

- We can also estimate the probability $p(e_1 \& e_2)$ that both experts were right.
- Based on all this information, we want to estimate the probability $p(E)$. 
7. What if we have only one expert?

- In this case, we know the conditional probability \( p_1 = p(E | e_1) \) and the probability \( p(e_1) \).
- We want to estimate the probability \( p(E) \).
8. Second problem

- In the second problem, we only consider folks who belong to one (or both) of the two populations.
- Let $E$ be equal to 1 if a randomly selected element has the desired phenomenon.
- Let $e_1$ be 1 if a randomly selected element belongs to the first population.
- Let $e_2$ be 1 if a randomly selected element belongs to the first population.
- We know the frequencies $p(E \mid e_1) = p_1$ and $p(E \mid e_2) = p_2$ of the phenomenon in each population.
- We know:
  - the number of elements $n_1$ in the first population,
  - the number of elements $n_2$ in the second population, and
  - the number of elements $n_{12}$ that belong to both populations.
9. Second problem (cont-d)

- In this case, the overall number of elements that belongs to both populations is equal to $n_1 + n_2 - n_{12}$.

- Thus, we can estimate the probabilities of $e_1$, $e_2$, and $e_1 \& e_2$ as follows:

\[
p(e_1) = \frac{n_1}{n_1 + n_2 - n_{12}}; \quad p(e_2) = \frac{n_2}{n_1 + n_2 - n_{12}}; \quad p(e_1 \& e_2) = \frac{n_{12}}{n_1 + n_2 - n_{12}}.
\]

- Based on all this information, we want to find the probability $p(E \mid e_1 \& e_2)$. 
10. Let us use the Maximum Entropy approach

- Situations in which we only have partial information about probabilities are ubiquitous.

- In such situations, several different probability distributions are consistent with our knowledge.

- In such cases, it makes sense not to pretend that our uncertainty is low.

- Thus, we should select the distribution with the largest possible uncertainty.

- A natural measure of uncertainty of a probability distribution is:
  - the average number of binary ("yes" - "no") questions that we need to ask
  - to fully determine which statements are true and which are false.
11. Let us use the Maximum Entropy approach (cont-d)

- It is known that this number is equal to Shannon’s entropy
  \[ S = - \sum P_i \cdot \log_2(P_i). \]
- Here \( P_i \) are the probabilities of different possible situations.
- Thus, we need to select the distribution with the largest possible entropy.
- Such a selection is known as the Maximum Entropy approach.
12. What this means for the first problem

- In the first problem, we have three basic statement $E$, $e_1$, and $e_2$.
- Each of these statements is either true or false.
- Thus, we have $2^3 = 8$ possible situations:
  
  $E \& e_1 \& e_2$,  $E \& e_1 \& \neg e_2$,  $E \& \neg e_1 \& e_2$,  $E \& \neg e_1 \& \neg e_2$,
  
  $\neg E \& e_1 \& e_2$,  $\neg E \& e_1 \& \neg e_2$,  $\neg E \& \neg e_1 \& e_2$,  $\neg E \& \neg e_1 \& \neg e_2$.

- Let us use the following notations for their probabilities:
  
  $p_{111} \overset{\text{def}}{=} E \& e_1 \& e_2$,  $p_{110} \overset{\text{def}}{=} E \& e_1 \& \neg e_2$,  $p_{101} \overset{\text{def}}{=} E \& \neg e_1 \& e_2$,
  
  $p_{100} \overset{\text{def}}{=} E \& \neg e_1 \& \neg e_2$,  $p_{011} \overset{\text{def}}{=} \neg E \& e_1 \& e_2$,  $p_{010} \overset{\text{def}}{=} \neg E \& e_1 \& \neg e_2$,
  
  $p_{001} \overset{\text{def}}{=} \neg E \& \neg e_1 \& e_2$,  $p_{000} \overset{\text{def}}{=} \neg E \& \neg e_1 \& \neg e_2$.

- These eight probabilities must add up to 1:
  
  $p_{111} + p_{110} + p_{101} + p_{100} + p_{011} + p_{010} + p_{001} + p_{000} = 1$.

- We know the probability $p(e_1)$.
13. What this means for the first problem (cont-d)

- Thus, the fact that we know the value $p_1 = p(E | e_1) = p(E & e_1)/p(e_1)$ is equivalent to knowing the probability
  \[ p(E & e_1) = p_1 \cdot p(e_1). \]

- In terms of the basic probabilities, the probability $p(E & e_1)$ has the form $p(E & e_1) = p(E & e_1 & e_2) + p(E & e_1 & \neg e_2) = p_{111} + p_{110}$.

- Thus, we have $p_{111} + p_{110} = p_1 \cdot p(e_1)$.

- Similarly, we know the probability $p(e_2)$.

- Thus, the fact that we know the value $p_2 = p(E | e_2) = p(E & e_2)/p(e_2)$ is equivalent to knowing the probability
  \[ (E & e_2) = p_2 \cdot p(e_2). \]

- In terms of the basic probabilities, the probability $p(E & e_2)$ has the form $p(E & e_2) = p(E & e_1 & e_2) + p(E & \neg e_1 & e_2) = p_{111} + p_{101}$.

- Thus, we have $p_{111} + p_{101} = p_2 \cdot p(e_2)$. 
14. What this means for the first problem (cont-d)

- Information about the values $p(e_1)$, $p(e_2)$, and $p(e_1 \& e_2)$ takes the following form: $p_{111} + p_{110} + p_{011} + p_{010} = p(e_1)$;
  
  $p_{111} + p_{101} + p_{011} + p_{001} = p(e_2)$;  $p_{111} + p_{011} = p(e_1 \& e_2)$.

- So, to find the values $p_{ikj}$, we need to maximize – under the above constraints – the entropy

  \[
  S = -p_{111} \cdot \log_2(p_{111}) - p_{110} \cdot \log_2(p_{110}) - p_{101} \cdot \log_2(p_{101}) - p_{100} \cdot \log_2(p_{100}) - p_{011} \cdot \log_2(p_{011}) - p_{010} \cdot \log_2(p_{010}) - p_{001} \cdot \log_2(p_{001}) - p_{000} \cdot \log_2(p_{000}).
  \]

- Entropy is a convex function of the probabilities, and the constraints are linear in terms of these probabilities.

- Thus, we can use the feasible convex optimization algorithms to find the desired probabilities.

- Once we find all the probabilities $p_{ijk}$, we can compute the desired probability $p(E)$ as $p(E) = p_{111} + p_{110} + p_{101} + p_{100}$.
15. Comments

- We can similarly consider the case when we have more than two experts and the case when we only have one expert.

- In the situation when we have only one expert, we have four possible situations $E \& e_1$, $E \& \neg e_1$, $\neg E \& e_1$, $\neg E \& \neg e_1$.

- Let us denote the probabilities of these situations by $p_{11} = p(E \& e_1)$, $p_{10} = p(E \& \neg e_1)$, $p_{01} = p(\neg E \& e_1)$, $p_{00} = p(\neg E \& \neg e_1)$.

- These probabilities must add up to 1: $p_{11} + p_{10} + p_{01} + p_{00} = 1$.

- The available information -- $p(e_1)$ and $p_1$ -- lead to the constraints $p_{11} + p_{01} = p(e_1)$ and $11 = p_1 \cdot p(e_1)$.

- We can thus determine the probability $p_{01}$ as

$$p_{01} = p(e_1) - p_1 \cdot p(e_1) = p(e_1) \cdot (1 - p_1).$$

- The only constraint on the remaining two values $p_{00}$ and $p_{10}$ is that all probabilities should add up to 1, so:

$$p_{00} + p_{10} = 1 - (p_{01} + p_{11}) = 1 - p(e_1).$$
16. Comments (cont-d)

- In this case, the maximum entropy approach leads to equal values of these two probabilities: $p_{00} = p_{10} = \frac{1 - p(e_1)}{2}$.

- Thus, the resulting estimate for the desired probability $p(E) = p_{11} + p_{10}$ has the form

$$p(E) = p_1 \cdot p(e_1) + \frac{1 - p(e_1)}{2} = \frac{1}{2} + p(e_1) \cdot \left( p_1 - \frac{1}{2} \right).$$

- This formula can be alternatively reformulated as

$$p(E) - \frac{1}{2} = p(e_1) \cdot \left( p_1 - \frac{1}{2} \right).$$

- In other words:
  - we should not take the expert estimate $p_1$ at face value,
  - we should adjust this estimate based on the expert’s track record.
17. What this means for the second problem

• From the mathematical viewpoint, the two problems has similar inputs.

• The only two differences are as follows:
  
  – first, we only consider objects that belong to one of the populations, so
    \[ p_{000} = p_{100} = 0; \]
  
  – second, what we want to estimate is different: \( p(E \mid e_1 \& e_2) \).

• Thus, to solve the second problem, we:
  
  – perform the same optimization as in the first problem – with the additional constraint \( p_{000} = p_{100} = 0; \)
  
  – thus, we find the probabilities \( p_{ijk} \), and then estimate:
    \[
    p(E \mid e_1 \& e_2) = \frac{p(E \& e_1 \& e_2)}{p(e_1 \& e_2)} = \frac{p_{111}}{p_{011} + p_{111}}.
    \]
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